UNIT-2

Lecture- 9

Arithmetic Progression(A.P) and its general term

ARITHMETIC PROGRESSION

SEQUENCE *A succession of numbers arranged in a definite order according to a certain given rule is called a sequence.*

The number occurring at the nth place of a sequence is called its *nth term* or the *general term*, to be denoted by a_n.

A sequence is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

By adding the terms of a sequence, we get a series.

A series is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

EXAMPLE 1 *Write first five terms of the sequence given by the rule* $a_n = (2n + 1)$ *and obtain the corresponding series.*

SOLUTION We have, $a_n = (2n + 1)$.

Putting $n = 1, 2, 3, 4, 5, \ldots$ successively in (i), we get

$$a_1 = (2 \times 1 + 1) = 3; a_2 = (2 \times 2 + 1) = 5; a_3 = (2 \times 3 + 1) = 7;$$

$$a_4 = (2 \times 4 + 1) = 9 \quad \text{and} \quad a_5 = (2 \times 5 + 1) = 11.$$

Hence, the required sequence is 3, 5, 7, 9, 11, ...

The corresponding series is $3 + 5 + 7 + 9 + 11 + \ldots$.

EXAMPLE 2 *Write first four terms of the sequence given by* $a_n = \frac{1}{6}(2n-3)$ *and obtain the corresponding series.*

EXAMPLE 3 *The Fibonacci sequence is defined by*

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2}, n > 2.$$

$$\text{Find } \frac{a_{n+1}}{a_n} \text{ for } n = 1, 2, 3, 4, 5.$$

PROGRESSIONS

Sequences following certain patterns are called progressions.

ARITHMETIC PROGRESSION (AP)

It is a sequence in which each term except the first one differs from its preceding term by a constant.

This constant difference is called the common difference of the AP.

In an AP we usually denote the *first term* by a, the *common difference* by d and the *nth term* by T_n.

Progressions: Sequence following certain patterns are called Progressions.

Types of Progressions:

(i) Arithmetic Progression (A.P)

(ii) Geometric Progression (G.P)

(i) Arithmetic Progression (A.P) - It is a sequence in which each term except the first one differs from its preceding term by a constant.

This constant difference is called the common difference of the A.P.

In A.P, firstterm = a

common difference = d

General term of an A.P :-

The n^{th} term of an A.P with first term a and common difference d is given by

$$T_n = a + (n-1)d$$

EXAMPLE 1 *Show that the sequence defined by* $T_n = 3n + 5$ *is an AP. Find its common difference.*

SOLUTION We have, $T_n = 3n + 5.$ ... (i)

Replacing n by $(n-1)$ in (i), we get

$$T_{n-1} = 3(n-1) + 5 \Rightarrow T_{n-1} = 3n + 2. \quad \text{... (ii)}$$

Subtracting (ii) from (i), we get

$$(T_n - T_{n-1}) = (3n+5) - (3n+2) = 3, \text{ which is constant.}$$

Hence, the given sequence is an AP with common difference 3.

EXAMPLE 2 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots$ *forms an AP. Find its common difference.*

SOLUTION By symmetry, we find that

$$T_n = \log\left(\frac{a^n}{b^{n-1}}\right) \text{ and } T_{n-1} = \log\left(\frac{a^{n-1}}{b^{n-2}}\right).$$

$$\therefore \quad (T_n - T_{n-1}) = \log\left(\frac{a^n}{b^{n-1}}\right) - \log\left(\frac{a^{n-1}}{b^{n-2}}\right)$$

$$= \log\left(\frac{a^n}{b^{n-1}} \times \frac{b^{n-2}}{a^{n-1}}\right) = \log\left(\frac{a}{b}\right) = \text{constant.}$$

Hence, the given sequence in an AP with common difference $\log\left(\frac{a}{b}\right)$.

EXAMPLE 3 *Show that the sequence defined by* $T_n = 3n^2 + 2$ *is not an AP.*

THEOREM 1 *Show that the nth term of an AP with first term a and common difference d is given by* $T_n = a + (n-1)d$.

In an AP with first term = a and common difference = d, we have

$$T_n = a + (n-1)d.$$

SOME SIMPLE PROPERTIES OF AN AP

(i) If a constant is added to each term of an AP then the resulting progression is an AP.

(ii) If a constant is subtracted from each term of an AP then the resulting progression is an AP.

(iii) If each term of an AP is multiplied by the same nonzero number then the resulting progression is an AP.

(iv) If each term of an AP is divided by the same nonzero number then the resulting progression is an AP.

EXAMPLE 1 *Show that the progression 7, 12, 17, 22, 27, … is an AP. Find its general term and the 14th term.*

Sol. We have, $T_n = 3n+5 \longrightarrow (1)$

So we have T_{n-1}, by replacing n by $(n-1)$ in (1)

we get

$$T_{n-1} = 3(n-1)+5 = 3n+2 \longrightarrow (2)$$

then $T_n - T_{n-1} = (3n+5) - (3n+2) = 3$

which is constant.

Hence the given sequence is an A.P

with common difference 3.

Q-1 Which term of the A.P: 3, 8, 13 - - - is 248? [2011-12, 2018-19]

Sol. In the given A.P,

$a = 3$, $d = 8 - 3 = 5$

So n^{th} term of A.P = 248

$$T_n = 248$$

$$a + (n-1)d = 248$$

$$3 + (n-1)5 = 248$$

$$5(n-1) = 248 - 3$$

$$5(n-1) = 245$$

$$5(n-1) = 248-3$$

$$5(n-1) = 245$$

$$(n-1) = 49$$

$$n = 50$$

So 50th term of the given A.P is 248.

Q-2 Find the n^{th} term of an A.P: 5, 8, 11 - -

Sol. In the given A.P, [2015-16]

$$a = 5, \ d = 3$$

$$\text{So } T_n = 5 + (n-1)3$$

$$= 5 + 3n - 3$$

$$\boxed{T_n = 3n + 2}$$

Q-3 Is 184 a term of the sequence 3, 7, 11 - - ?

[2019-20]

Sol. In the given A.P,

$a = 3, \; d = 7 - 3 = 4$

$T_n = 184$

$a + (n-1)d = 184$

$3 + (n-1)4 = 184$

$3 + 4n - 4 = 184$

$4n - 1 = 184$

$4n = 185$

$$4n - 1 = 184$$

$$4n = 185$$

$$n = 46.25 = 46\frac{1}{4}$$

Since the number of the terms cannot be a fraction, so it follows that 184 is not a term of the given A.P.

Q-4 Which term of the A·P:

$3, 8, 13 - - \text{ is } 78$? [2020-21]

Sol· In the given A·P,

$a = 3, \ d = 8 - 3 = 5$

$T_n = 78$

$$a + (n-1)d = 78$$

$$3 + (n-1)5 = 78$$

$$3 + 5n - 5 = 78$$

$$5n - 2 = 78$$

$$5n = 80$$

$$n = 16$$

Hence the 16th term of the given A.P is

78.

Q-5 If $a_n = 5 - 11n$, find the common difference. [2020-21]

Sol.

$$a_n = 5 - 11n$$

$$a_{n-1} = 5 - 11(n-1)$$

$$= 5 - 11n + 11$$

$$a_{n-1} = 16 - 11n$$

$$a_{n-1} = 16 - 11n$$

So

$$d = a_n - a_{n-1} = 16 - 11n - 5 + 11n$$

$$d = 11$$

Hence common difference $= 11$.

n^{th} term from the end of an A.P :-

The n^{th} term from the end of an A.P with the first term a and common difference d and the last term l is given by

$$T_n = l - (n-1)d$$

PRACTICE QUESTIONS

1. Write first 4 terms in each of the sequences:

   (i) $a_n = (5n+2)$ (ii) $a_n = \frac{(2n-3)}{4}$ (iii) $a_n = (-1)^{n-1} \times 2^{n+1}$

2. Find first five terms of the sequence, defined by

$$a_1 = 1, a_n = a_{n-1} + 3 \text{ for } n \geq 2.$$

3. Find first 5 terms of the sequence, defined by

$$a_1 = -1, a_n = \frac{a_{n-1}}{n} \text{ for } n \geq 2.$$

4. Find the 23rd term of the AP 7, 5, 3, 1, –1, –3, ...
5. Find the 20th term of the AP $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \ldots$.
6. Find the nth term of the AP 8, 3, –2, –7, –12, ...
7. Find the nth term of the AP $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \ldots$.

PRACTICE QUESTIONS

8. Which term of the AP 9, 14, 19, 24, 29, … is 379?
9. Which term of the AP 64, 60, 56, 52, 48, … is 0?
10. How many terms are there in the AP 11, 18, 25, 32, 39, …, 207?
11. How many terms are there in the AP $1\frac{5}{6}, 1\frac{1}{6}, \frac{1}{2}, \frac{-1}{6}, \frac{-5}{6}, \ldots, -16\frac{1}{6}$?

ANSWERS

1. (i) $7, 12, 17, 22$ (ii) $\frac{-1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$ (iii) $4, -8, 16, -32$ 2. $1, 4, 7, 10, 13$

3. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120}$ 4. -37 5. $39\sqrt{2}$ 6. $T_n = (13 - 5n)$

7. $T_n = \frac{1}{6}(7 - n)$ 8. 75th 9. 17th 10. 29 11. 28

- ***THANK YOU***

Lecture- 10,11

Sum of n^{th} terms of a A.P :- The sum of n terms of an A.P with first term a and common difference d is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

If l is the last term then

$$S_n = \frac{n}{2}(a + l)$$

① Find the sum of 23 terms of the AP 5, 9, 13, 17 ....

Sol. Here $a = 5$, $d = 9 - 5 = 4$, $n = 23$

Now, $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{23}{2}[2 \times 5 + (23-1) \times 4]$

$= \frac{23}{2} \times 98 = 1127$ Ans.

(2) Find the sum of all odd integers from 1 to 1001

Sol. The odd integers from 1 to 1001 are 1, 3, 5 - - 999, 1001,

This is an A.P in which $a = 1$, $d = 3 - 1 = 2$, $l = 1001$, let n be the number of terms.

$T_n = a + (n-1)d$

$$T_n = a + (n-1)d$$

$$1001 = 1 + (n-1) \times 2$$

$$\Rightarrow \quad n = 501$$

Now $a = 1$, $l = 1001$ & $n = 501$

$$S_n = \frac{n}{2}(a+l) = \frac{501}{2} \cdot (1 + 1001) = 501 \times 501$$

$$= 251001.$$

Q-3 The third term of an A.P is 7 and the seventh term is 2 more than 3 times the third term. Find the first term, the common difference and the sum of first 20 terms. [2011-12]

Sol.

$a_3 = T_3 = 7, \quad T_7 = 3T_3 + 2, \quad a = ?$

$d = ?$

$T_3 = a + (n-1)d = 7$

Sol.

$a_3 = T_3 = 7, \quad T_7 = 3T_3 + 2, \quad a = ?$

$d = ?$

$S_{20} = ?$

$T_3 = a + (n-1)d = 7$

$a + (3-1)d = 7$

$a + 2d = 7 \longrightarrow (1)$

$T_7 = 3T_3 + 2$

$a + (7-1)d = 3[a + 2d] + 2$

$T_7 = 3T_3 + 2$

$a + (7-1)d = 3[a + 2d] + 2$

$a + 6d = 3a + 6d + 2$

$\Rightarrow \quad 2a = 2$

$\Rightarrow \quad \boxed{a = 1}$

Use in (1);

$1 + 2d = 7$

$2d = 7 - 1$

$2d = 6$

$\boxed{d = 3}$

$$S_{20} = \frac{20}{2}[2 \times 1 + (20-1)3]$$

$$= 10[2 + 19 \times 3]$$

$$= 10[57 + 2]$$

$$= 10 \times 59$$

$$\boxed{S_{20} = 590} \text{ Ans}$$

Q.7 Find the sum of $2+4+6+\dots+20$

$$a=2,\ l=20,\ d=2$$

$$S_n=\frac{n}{2}[a+l] \longrightarrow (1)$$

$$T_n=a+(n-1)d$$

$$20=2+(n-1)2$$

$$10=1+n-1$$

$$n=10$$

By (1)

$$S_n = \frac{10}{2}[2+20] = 110 \text{ Ans}$$

Q8 Find the sum of $3+6+9+ --- +30$.

Sol. $a=3,\ T_n=30$ [2020-21]

$T_n = a+(n-1)d = 3+(n-1)\times 3 = 30$

$\Rightarrow n=10$

So $S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{10}{2}[6+(10-1)3] = 5\times 33$

$= 165$ Ans

Q-9 The first term of an A.P is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

[2020-21]

Sol. Given

$a = 5,\ l = 45,\ S_n = 400,\ T_n = 45$

$S_n = \frac{n}{2}[a+l] \Rightarrow \frac{n}{2}(50) = 400$

$$n = \frac{800}{50} = 16$$

So number of terms = 16

Common difference d = ?

$$T_n = a + (n-1)d$$

$$T_{16} = 5 + (16-1)d$$

$$45 = 5 + 15d$$

$$40 = 15d$$

$$\boxed{d = 8/3}$$ Ans

PRACTICE QUESTIONS

1. Find the sum of 23 terms of the AP 17, 12, 7, 2, –3, … .
2. Find the sum of 16 terms of the AP $6, 5\frac{1}{3}, 4\frac{2}{3}, 4, \ldots$.
3. Find the sum of 25 terms of the AP $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots$.
4. Find the sum of 100 terms of the AP 0.6, 0.61, 0.62, 0.63, … .
5. Find the sum of 20 terms of the AP $(x+y), (x-y), (x-3y), \ldots$.
6. Find the sum of n terms of the AP $\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y}, \ldots$.
7. Find the sum of the series $2+5+8+11+\ldots+191$.
8. Find the sum of the series $101+99+97+95+\ldots+43$.

ANSWERS

1. -874
2. 16
3. $325\sqrt{2}$
4. 109.5
5. $20(x-18y)$
6. $\frac{n}{2(x+y)}\cdot|n(2x-y)-y|$
7. 6176
8. 2160

- ***THANK YOU***

Lecture- 12

Arithmetic Mean

ARITHMETIC MEAN

ARITHMETIC MEAN

If a, A, b are in AP then we say that A is the arithmetic mean (AM) between a and b.

INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN a AND b

Let a and b be two given numbers and let A be the arithmetic mean between a and b. Then,

a, A, b are in AP

$\Rightarrow \quad A - a = b - A$

$\Rightarrow \quad 2A = a + b \Rightarrow A = \frac{a+b}{2}$.

Hence, the arithmetic mean between a and b is $\frac{a+b}{2}$.

EXAMPLE 1 *Find the arithmetic mean between*

(i) 14 *and* –6, *(ii)* $(a-b)$ *and* $(a+b)$.

SOLUTION (i) Arithmetic mean between 14 and –6

$$= \frac{14+(-6)}{2} = \frac{8}{2} = 4.$$

(ii) Arithmetic mean between $(a-b)$ and $(a+b)$

$$= \frac{(a-b)+(a+b)}{2} = \frac{2a}{2} = a.$$

Q-2 Insert six arithmetic means between 15 and -13.

Sol. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be the six arithmetic means between 15 and -13 then

So 15, A_1, A_2, A_3, A_4, A_5, A_6, −13 are in A.P

So $d = \frac{(-13-15)}{(6+1)} = -4$

So $A_1 = (15+d) = 15-4 = 11$, $A_2 = (15+2d) = 7$

$A_3 = (15+3d) = 15-12 = 3$

$A_4 = (15+4d) = 15-16 = -1$

$A_5 = (15+5d) = 15-20 = -5$

$A_6 = (15+6d) = 15-24 = -9$

Hence the required six AMs between 15 and −13 is

11, 7, 3, −1, −5, −9.

Some more question related to A.P -

① If a, b, c are in A.P, show that

(i) (b+c), (c+a) and (a+b) are in A.P

Sol. Since a, b, c are in A.P we have

$$2b = a + c \longrightarrow (1)$$

(i) (b+c), (c+a), (a+b) will be in A.P if

$$(c+a)-(b+c) = (a+b)-(c+a)$$

So if $a - b = b - c$

So if $a-b = b-c$

$$2b = a+c$$

which is true by (i)

Hence $(b+c), (c+a), (a+b)$ are in A.P.

PRACTICE QUESTIONS

1. Find the arithmetic mean between
   (i) 9 and 19, (ii) 15 and –7, (iii) –16 and –8.
2. Insert four arithmetic means between 4 and 29.
3. Insert three arithmetic means between 23 and 7.
4. Insert six arithmetic means between 11 and –10.
5. There are n arithmetic means between 9 and 27. If the ratio of the last mean to the first mean is 2 : 1, find the value of n.
6. Insert arithmetic means between 16 and 65 such that the 5th AM is 51. Find the number of arithmetic means.
7. Insert five numbers between 11 and 29 such that the resulting sequence is an AP.

ANSWERS

1. (i) 14 (ii) 4 (iii) –12
2. 9, 14, 19, 24
3. 19, 15, 11
4. 8, 5, 2, –1, –4, –7
5. $n = 5$
6. six
7. 14, 17, 20, 23, 26

- ***THANK YOU***

Lecture- 12

Geometric Progression(G.P) and its general term

GEOMETRICAL PROGRESSION (GP)

A sequence $a_1, a_2, a_3, \ldots, a_n$ is called a geometrical progression, if each term is nonzero and $\frac{a_{k+1}}{a_k} = r$ *(constant) for all* $k \geq 1$.

The constant ratio is called its *common ratio.*

A geometrical progression is abbreviated as GP.

In a GP we usually denote the *first term* by *a*, the *common ratio* by *r* and the *nth term* by T_n.

The *n*th term of a GP is called its general term.

GENERAL TERM OF A GP

REMEMBER

In a GP with first term = a and common ratio = r, we have

$$n\text{th term, } T_n = ar^{n-1}.$$

Example Find the 10^{th} and n^{th} terms of the G.P. 5, 25,125,.... .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10-1} = 5(5)^9 = 5^{10}$
and $a_n = ar^{n-1} = 5(5)^{n-1} = 5^n$.

Example Which term of the G.P., 2,8,32, ... up to n terms is 131072?

Solution Let 131072 be the n^{th} term of the given G.P. Here $a = 2$ and $r = 4$.

Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$

This gives $4^8 = 4^{n-1}$.

So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9^{th} term of the G.P.

Example : In a G.P., the 3rd term is 24 and the 6th term is 192.Find the 10th term.

Solution Here, $a_3 = ar^2 = 24$ ... (1)

and $a_6 = ar^5 = 192$ ... (2)

Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.

Hence $a_{10} = 6\,(2)^9 = 3072$.

EXAMPLE *Show that the progression* 6, 18, 54, 162, ... *is a GP. Write down its first term and the common ratio.*

SOLUTION We have $\frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3$ (constant).

So, the given progression is a GP in which the first term = 6 and the common ratio = 3.

EXAMPLE. *Show that the progression* $-16, 4, -1, \frac{1}{4}, \ldots$ *is a GP. Write down its first term and the common ratio.*

SOLUTION We have $\frac{4}{-16} = \frac{-1}{4} = \frac{(1/4)}{-1} = \frac{-1}{4}$ (constant).

So, the given progression is a GP in which $a = -16$ and $r = \frac{-1}{4}$.

EXAMPLE : *Find the 10th term and the general term of the progression*

$$\frac{1}{4}, \frac{-1}{2}, 1, -2, 4, \ldots .$$

SOLUTION In the given progression, we have

$$\left(\frac{-1}{2}\right) \div \frac{1}{4} = \left(\frac{-1}{2} \times 4\right) = -2,\ 1 \div \left(\frac{-1}{2}\right) = 1 \times (-2) = -2,$$

$$(-2) \div 1 = -2 \text{ and } 4 \div (-2) = -2.$$

So, the given progression is a GP in which $a = \frac{1}{4}$ and $r = -2$.

$\therefore$ the 10th term, $T_{10} = ar^{(10-1)} = ar^9 = \frac{1}{4} \times (-2)^9 = \frac{-512}{4} = -128.$

The general term, $T_n = ar^{(n-1)} = \frac{1}{4} \times (-2)^{(n-1)} = (-1)^{(n-1)} \times 2^{(n-3)}.$

EXAMPLE *Show that the progression*

$$1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \frac{(3-2\sqrt{2})}{12}, \frac{(5\sqrt{2}-7)}{24\sqrt{3}}, \dots$$

is a GP. Find its 5th term.

SOLUTION In the given progression, we have

$$\frac{T_2}{T_1} = \left\{\frac{(\sqrt{2}-1)}{2\sqrt{3}} \times \frac{1}{1}\right\} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \quad \frac{T_3}{T_2} = \frac{(3-2\sqrt{2})}{12} \times \frac{2\sqrt{3}}{(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)}{2\sqrt{3}},$$

$$\frac{T_4}{T_3} = \frac{(5\sqrt{2}-7)}{24\sqrt{3}} \times \frac{12}{(3-2\sqrt{2})} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}.$$

$$\therefore \quad \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \ldots = \frac{(\sqrt{2}-1)}{2\sqrt{3}} \text{ (constant).}$$

So, the given progression is a GP in which $a = 1$ and $r = \frac{(\sqrt{2}-1)}{2\sqrt{3}}$.

$$\therefore \quad \text{the 5th term, } T_5 = ar^{(5-1)} = ar^4 = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4$$

$$= \frac{(\sqrt{2}-1)^4}{144} = \frac{(3-2\sqrt{2})^2}{144}$$

$$= \frac{(17-12\sqrt{2})}{144}.$$

Hence, $T_5 = \frac{(17-12\sqrt{2})}{144}$.

EXAMPLE *The 4th, 7th and 10th terms of a GP are a, b, c respectively. Prove that $b^2 = ac$.*

SOLUTION Let A be the first term and r be the common ratio of the given GP. Then,

$$a = Ar^{(4-1)} = Ar^3; b = Ar^{(7-1)} = Ar^6 \text{ and } c = Ar^{(10-1)} = Ar^9.$$

$$\therefore \quad ac = (Ar^3) \times (Ar^9) = A^2 r^{12} = (Ar^6)^2 = b^2.$$

Hence, $b^2 = ac$.

EXAMPLE *If a, b, c are three consecutive terms of an AP and x, y, z are three consecutive terms of a GP, then prove that*

$$x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)} = 1.$$

SOLUTION It is given that a, b, c are in AP. Let d be the common difference of this AP. Then,

$$(b-c) = -(c-b) = -d, \ (c-a) = \{(c-b)+(b-a)\} = 2d$$

and $(a-b) = -(b-a) = -d.$

Also, x, y, z are in GP. So, $y = \sqrt{xz}$.

$$\therefore \quad x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)}$$

$$= x^{-d} \times (\sqrt{xz})^{2d} \times z^{-d}$$

$$[\because (b-c) = -d, (c-a) = 2d, (a-b) = -d \text{ and } y = \sqrt{xz}]$$

$$= x^{-d} \times (xz)^{d} \times z^{-d}$$

$$= x^{-d} \times x^{d} \times z^{d} \times z^{-d} = x^{(-d+d)} \times z^{(-d+d)} = (x^{0} \times z^{0}) = 1.$$

Hence, $x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)} = 1.$

EXAMPLE ' _ *If a, b, c are in GP and* $a^{1/x} = b^{1/y} = c^{1/z}$*, prove that x, y, z are in AP.*

SOLUTION Since a, b, c are in GP, we have

$$b^2 = ac. \quad \ldots$$

Let $a^{1/x} = b^{1/y} = c^{1/z} = k$ (say).

Then, $a = k^x$, $b = k^y$ and $c = k^z$.

Putting these values in (i), we get

$$(k^y)^2 = (k^x) \times (k^z) \Rightarrow k^{2y} = k^{(x+z)}.$$

$$\therefore \quad 2y = x + z.$$

Hence, x, y, z are in AP.

n^{th} term from the end of a G.P :

The n^{th} term from the end of a G.P with the first term a, the common ratio r and the last term l is given by

$$\frac{l}{r^{n-1}}.$$

EXAMPLE *Show that the progression*

$$1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \frac{(3-2\sqrt{2})}{12}, \frac{(5\sqrt{2}-7)}{24\sqrt{3}}, \ldots$$

is a GP. *Find its 5th term.*

SOLUTION In the given progression, we have

$$\frac{T_2}{T_1} = \left\{\frac{(\sqrt{2}-1)}{2\sqrt{3}} \times \frac{1}{1}\right\} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \quad \frac{T_3}{T_2} = \frac{(3-2\sqrt{2})}{12} \times \frac{2\sqrt{3}}{(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)}{2\sqrt{3}},$$

$$\frac{T_4}{T_3} = \frac{(5\sqrt{2}-7)}{24\sqrt{3}} \times \frac{12}{(3-2\sqrt{2})} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}.$$

$$\therefore \quad \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \dots = \frac{(\sqrt{2}-1)}{2\sqrt{3}} \text{ (constant).}$$

So, the given progression is a GP in which $a = 1$ and $r = \frac{(\sqrt{2}-1)}{2\sqrt{3}}$.

$$\therefore \quad \text{the 5th term, } T_5 = ar^{(5-1)} = ar^4 = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4$$

$$= \frac{(\sqrt{2}-1)^4}{144} = \frac{(3-2\sqrt{2})^2}{144}$$

$$= \frac{(17-12\sqrt{2})}{144}.$$

Hence, $T_5 = \frac{(17-12\sqrt{2})}{144}$.

PRACTICE QUESTIONS

1. Find the 6th and nth terms of the GP $2, 6, 18, 54, \ldots$.
2. Find the 17th and nth terms of the GP $2, 2\sqrt{2}, 4, 8\sqrt{2}, \ldots$.
3. Find the 7th and nth terms of the GP $0.4, 0.8, 1.6, \ldots$.
4. Find the 10th and nth terms of the GP $\frac{-3}{4}, \frac{1}{2}, \frac{-1}{3}, \frac{2}{9}, \ldots$.
5. Which term of the GP $3, 6, 12, 24, \ldots$ is 3072?
6. Which term of the GP $\frac{1}{4}, \frac{-1}{2}, 1, \ldots$ is -128?
7. Which term of the GP $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729?

PRACTICE QUESTIONS

8. In a GP, the ratio of the sum of first three terms is to that of first six terms is 125 : 152. Find the common ratio.
9. Find the sum of the geometric series $3 + 6 + 12 + \ldots + 1536$.

ANSWERS

1. $486, 2 \times 3^{(n-1)}$ 2. $512, (\sqrt{2})^{(n+1)}$ 3. $25.6, \frac{2^n}{5}$ 4. $\frac{128}{6561}, \frac{-3}{4} \times \left(\frac{-2}{3}\right)^{(n-1)}$

5. 11th 6. 10th 7. 12th

ANSWERS

8. $r = \frac{3}{5}$

9. 3069

- ***THANK YOU***

Lecture- 13

Sum of n terms of Geometric Progression(G.P) and its Infinite term

SUM OF n TERMS OF A GP

THEOREM *Prove that the sum of n terms of a GP with the first term a and the common ratio r is given by*

$$S_n = \begin{cases} na, \text{ when } r = 1; \\ \dfrac{a(1-r^n)}{(1-r)}, \text{ when } r < 1; \\ \dfrac{a(r^n-1)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

EXAMPLE 1 *Find the sum of 8 terms of the GP 3, 6, 12, 24, ...*

SOLUTION Here $a = 3$, $r = 2 > 1$ and $n = 8$.

Using the formula, $S_n = \frac{a(r^n - 1)}{(r-1)}$, we get

$$S_8 = \frac{3 \times (2^8 - 1)}{(2-1)} = 3 \times (256 - 1) = 3 \times 255 = 765.$$

EXAMPLE 2 *Find the sum of the geometric series* $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ *to 12 terms.*

SOLUTION Here $a=1$, $r=\frac{1}{2}<1$ and $n=12$.

Using the formula, $S_n=\frac{a(1-r^n)}{(1-r)}$, we get

$$S_{12}=\frac{1\times\left\{1-\left(\frac{1}{2}\right)^n\right\}}{\left(1-\frac{1}{2}\right)}=\frac{\left(1-\frac{1}{2^{12}}\right)}{\left(\frac{1}{2}\right)}=\frac{(2^{12}-1)}{2^{11}}=\frac{4095}{2048}.$$

EXAMPLE Find the sum of the series $2+6+18+54+\dots+4374$.

SOLUTION Clearly, the given series is a geometric series in which $a=2$, $r=3>1$ and $l=4374$.

$$\therefore \quad \text{the required sum} = \frac{(lr-a)}{(r-1)} = \frac{(4374\times 3-2)}{(3-1)} = \frac{13120}{2} = 6560.$$

Hence, the sum of the given series is 6560.

EAMPLE 5 *In a GP, it is being given that $T_1 = 3, T_n = 96$ and $S_n = 189$. Find the value of n.*

SOLUTION Here, $a = 3, l = 96$ and $S_n = 189$.

Let the common ratio of the given GP be r.

Then, $S_n = \frac{(lr-a)}{(r-1)} \Rightarrow \frac{(96r-3)}{(r-1)} = 189$

$$\Rightarrow (96r-3) = (189r-189)$$

$$\Rightarrow 93r = 186 \Rightarrow r = 2.$$

Now, $l = ar^{n-1} \Rightarrow 3 \times 2^{n-1} = 96$

$$\Rightarrow 2^{n-1} = 32 = 2^5 \Rightarrow n-1 = 5 \Rightarrow n = 6.$$

Hence, $n = 6$.

EXAMPLE *Sum the series $5+55+555+\dots$ to n terms.*

SOLUTION We have

$$5+55+555+\dots \text{ to } n \text{ terms}$$

$$= 5 \times \{1+11+111+\dots \text{ to } n \text{ terms}\}$$

$$= \frac{5}{9} \times \{9+99+999+\dots \text{ to } n \text{ terms}\}$$

$$= \frac{5}{9} \times \{(10-1)+(10^2-1)+(10^3-1)+\dots \text{ to } n \text{ terms}\}$$

$$= \frac{5}{9} \times \{(10+10^2+10^3+\dots \text{ to } n \text{ terms})-n\}$$

$$= \frac{5}{9} \times \left\{\frac{10 \times (10^n-1)}{(10-1)}-n\right\} = \frac{5}{81} \times (10^{n+1}-9n-10).$$

Hence, the required sum is $\frac{5}{81} \times (10^{n+1}-9n-10)$.

EXAMPLE. *Sum the series* $.4 + .44 + .444 + \ldots$ *to n terms.*

SOLUTION We have

$$.4 + .44 + .444 + \ldots \text{ to } n \text{ terms}$$

$$= 4 \times \{.1 + .11 + .111 + \ldots \text{ to } n \text{ terms}\}$$

$$= \frac{4}{9} \times \{.9 + .99 + .999 + \ldots \text{ to } n \text{ terms}\}$$

$$= \frac{4}{9} \times \{(1 - .1) + (1 - .01) + (1 - .001) + \ldots \text{ to } n \text{ terms}\}$$

$$= \frac{4}{9} \times \{(1 + 1 + \dots \text{ to } n \text{ terms}) - (.1 + .01 + .001 + \dots \text{ to } n \text{ terms})\}$$

$$= \frac{4}{9} \times \left[n - \frac{.1 \times \{1 - (.1)^n\}}{(1 - .1)} \right] \qquad \left\{ \because S_n = \frac{a(1 - r^n)}{(1 - r)} \right\}$$

$$= \frac{4}{9} \times \left[n - \frac{\frac{1}{10} \cdot \left\{ 1 - \frac{1}{(10)^n} \right\}}{\left(1 - \frac{1}{10} \right)} \right]$$

$$= \frac{4}{9} \times \left\{ n - \frac{(10^n - 1)}{9 \cdot 10^n} \right\} = \frac{4}{9} \times \left[n - \frac{1}{9} \left\{ 1 - \frac{1}{10^n} \right\} \right]$$

$$= \frac{4}{81} \times \left[9n - \left\{ 1 - \frac{1}{10^n} \right\} \right] = \frac{4}{81} \times \left\{ 9n - 1 + \frac{1}{10^n} \right\}.$$

Hence, the required sum is $\frac{4}{81} \times \left(9n - 1 + \frac{1}{10^n} \right)$.

SUMMARY

Sum of an infinite GP with the first term a and the common ratio r, where $|r| < 1$, is given by $S = \frac{a}{(1-r)}$.

EXAMPLE 1 *Find the sum of the infinite geometric series* $\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty\right)$.

SOLUTION In the given infinite geometric series, we have

$$a = 1 \text{ and } r = \frac{1}{3} \text{ such that } |r| = \frac{1}{3} < 1.$$

Hence, the sum of the given infinite series is

$$S = \frac{a}{(1-r)} = \frac{1}{\left(1-\frac{1}{3}\right)} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}.$$

Hence, the required sum is $\frac{3}{2}$.

EXAMPLE 2 *Find the sum of the infinite geometric series* $\left(1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\ldots\infty\right)$.

SOLUTION The given series is an infinite geometric series in which

$$a=1, r=-\frac{1}{3} \text{ and } |r|=\frac{1}{3}<1.$$

Hence, the sum of the given infinite geometric series is

$$S=\frac{a}{(1-r)}=\frac{1}{\left(1+\frac{1}{3}\right)}=\frac{1}{\left(\frac{4}{3}\right)}=\frac{3}{4}.$$

EXAMPLE *Find the sum of the infinite geometric series*

$$(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots\infty.$$

SOLUTION We have

$$\frac{1}{(\sqrt{2}+1)}=\frac{1}{(\sqrt{2}+1)}\times\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}=\frac{(\sqrt{2}-1)}{1}.$$

So, the given series is an infinite geometric series in which $a=(\sqrt{2}+1)$ and $r=(\sqrt{2}-1)<1$.

Hence, the sum of the given infinite geometric series is

$$S=\frac{a}{(1-r)}=\frac{(\sqrt{2}+1)}{\{1-(\sqrt{2}-1)\}}=\frac{(\sqrt{2}+1)}{(2-\sqrt{2})}\times\frac{(2+\sqrt{2})}{(2+\sqrt{2})}$$

$$=\frac{4+3\sqrt{2}}{(4-2)}=\frac{(4+3\sqrt{2})}{2}.$$

EXAMPLE Prove that $6^{1/2} \cdot 6^{1/4} \cdot 6^{1/8} \cdot \ldots \infty = 6.$

SOLUTION We observe here that $\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)$ is an infinite geometric series in which $a=\frac{1}{2}$ and $r=\left(\frac{1}{4}\times\frac{2}{1}\right)=\frac{1}{2}$ such that $|r|<1$.

So, this sum is given by

$$S=\frac{\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}=1. \qquad \ldots \text{(i)}$$

$$\therefore \quad 6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \ldots \infty = 6^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)} = 6^{1} = 6 \qquad \text{[using (i)].}$$

Hence, $6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \ldots \infty = 6.$

PRACTICE QUESTIONS

1. Find the sum of the GP:

   (i) $1 + 3 + 9 + 27 + \ldots$ to 7 terms

   (ii) $1 + \sqrt{3} + 3 + 3\sqrt{3} + \ldots$ to 10 terms

   (iii) $0.15 + 0.015 + 0.0015 + \ldots$ to 6 terms

   (iv) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ to 9 terms

   (v) $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \ldots$ to 8 terms

   (vi) $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - \frac{3}{4} + \ldots$ to 6 terms

2. Find the sum of the GP:

   (i) $\sqrt{7} + \sqrt{21} + 3\sqrt{7} + \ldots$ to n terms

   (ii) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots$ to n terms

   (iii) $1 - a + a^2 - a^3 + \ldots$ to n terms $(a \neq 1)$

PRACTICE QUESTIONS

Find the sum of the series:

(i) $8 + 88 + 888 + \ldots$ to n terms

(ii) $3 + 33 + 333 + \ldots$ to n terms

(iii) $0.7 + 0.77 + 0.777 + \ldots$ to n terms

ANSWERS

1. (i) 1093 (ii) $121(\sqrt{3}+1)$ (iii) $\frac{333333}{2000000}$ (iv) $\frac{171}{256}$

   (v) $\frac{255\sqrt{2}}{128}$ (vi) $\frac{-133}{144}$

2. (i) $\frac{\sqrt{7}}{2}(\sqrt{3}+1)(3^{n/2}-1)$ (ii) $\frac{\{3^n-(-1)^n\}}{4\times 3^{(n-1)}}$ (iii) $\frac{\{1-(-a)^n\}}{(1+a)}$

ANSWERS

(i) $\frac{8}{81}[10^{(n+1)} - 10 - 9n]$ (ii) $\frac{1}{27}[10^{(n+1)} - 10 - 9n]$ (iii) $\frac{7}{81}\left(9n - 1 + \frac{1}{10^n}\right)$

- ***THANK YOU***

Lecture- 14

Sum to n terms of Special Series

SUM OF FIRST *n* NATURAL NUMBERS

<u>THEOREM 1</u> *Prove that* $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$

Some Special Series :-

① $(1+2+3+ \cdots +n) = \frac{n}{2}(n+1)$

$$\sum_{n=1}^{n} n = \frac{1}{2}n(n+1)$$

② $(1^2+2^2+3^2+ \cdots +n^2) = \frac{n}{6}(n+1)(2n+1)$

$$\sum_{n=1}^{n^2} n^2 = \frac{n}{6}(n+1)(2n+1)$$

(3) $$(1^3+2^3+3^3+\dots+n^3) = \left[\frac{n}{2}(n+1)\right]^2$$

$$\sum_{n=1}^{n} n^3 = \left(\frac{n}{2}\right)^2 (n+1)^2$$

MIET GROUP OF INSTITUTIONS

EXAMPLE 1 *If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that $9S_2^2 = S_3(1+8S_1)$.*

SOLUTION We have

$$S_1 = (1+2+3+\ldots+n) \Rightarrow S_1 = \frac{1}{2}n(n+1);$$

$$S_2 = (1^2+2^2+3^2+\ldots+n^2) \Rightarrow S_2 = \frac{1}{6}n(n+1)(2n+1);$$

and $$S_3 = (1^2+2^3+3^3+\ldots+n^3) \Rightarrow S_3 = \frac{1}{4}n^2(n+1)^2.$$

$$\therefore \quad 9S_2^2 = 9 \times \frac{1}{36} \cdot n^2(n+1)^2(2n+1)^2 = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

$$\text{And, } S_3(1+8S_1) = \frac{1}{4}n^2(n+1)^2 \cdot \left\{1 + 8 \cdot \frac{1}{2}n(n+1)\right\}$$

$$= \frac{1}{4}n^2(n+1)^2(4n^2+4n+1) = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

$$\text{Hence, } 9S_2^2 = S_3(1+8S_1).$$

EXAMPLE *Find the sum to n terms of the series whose nth term is* $(2n-1)^2$.

SOLUTION We have, $T_k = (2k-1)^2 = (4k^2 - 4k + 1)$.

$\therefore$ sum to n terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (4k^2 - 4k + 1) = 4 \cdot \sum_{k=1}^{n} k^2 - 4 \cdot \sum_{k=1}^{n} k + n$$

$$[\because\ 1 + 1 + \ldots n \text{ times} = n]$$

$$= 4 \cdot \frac{1}{6} n(n+1)(2n+1) - 4 \cdot \frac{1}{2} n(n+1) + n$$

$$= \frac{2}{3} n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{1}{3} \cdot \{2n(n+1)(2n+1) - 6n(n+1) + 3n\}$$

$$= \frac{1}{3}[n(n+1)\{2(2n+1) - 6\} + 3n] = \frac{1}{3} \cdot [4n(n+1)(n-1) + 3n]$$

$$= \frac{1}{3}(4n^3 - n) = \frac{1}{3} n(4n^2 - 1).$$

Hence, the required sum is $\frac{1}{3} n(4n^2 - 1)$.

EXAMPLE *Find the sum of the series $1^2 + 3^2 + 5^2 + \ldots$ to n terms.*

SOLUTION We have

$$T_k = k\text{th term of } (1^2 + 3^2 + 5^2 + \ldots)$$
$$= \{1 + (k-1) \times 2\}^2 = (2k-1)^2 = (4k^2 - 4k + 1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} (4k^2 - 4k + 1)$$
$$= 4 \sum_{k=1}^{n} k^2 - 4 \sum_{k=1}^{n} k + (1 + 1 + \ldots n \text{ times})$$
$$= 4 \cdot \frac{1}{6} n(n+1)(2n+1) - 4 \cdot \frac{1}{2} n(n+1) + n$$
$$= \frac{n}{3} \cdot \{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}(4n^2 - 1).$$

Hence, the required sum is $\frac{n}{3}(4n^2 - 1)$.

EXAMPLE *Find the sum of n terms of the series*

$$\frac{1}{(2\times 5)}+\frac{1}{(5\times 8)}+\frac{1}{(8\times 11)}+\cdots.$$

SOLUTION We have

$$T_k = \frac{1}{(k\text{th term of } 2, 5, 8, \ldots) \times (k\text{th term of } 5, 8, 11, \ldots)}$$

$$= \frac{1}{\{2 + (k-1) \times 3\} \times \{5 + (k-1) \times 3\}}$$

$$= \frac{1}{(3k-1)(3k+2)} = \frac{1}{3}\left\{\frac{1}{(3k-1)} - \frac{1}{(3k+2)}\right\}.$$

$$\therefore \quad T_k = \frac{1}{3}\left\{\frac{1}{(3k-1)} - \frac{1}{(3k+2)}\right\}. \quad \ldots \text{(i)}$$

Putting $k = 1, 2, 3, \ldots, n$ successively in (i), we get

$$T_1 = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$T_2 = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right)$$

… … … …

… … … …

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)} - \frac{1}{(3n+2)}\right\}.$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)} - \frac{1}{(3n+2)}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \ldots + T_n)$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{n}{2(3n+2)}.$$

PRACTICE QUESTIONS

QUES-01

$$1 \times 2^2 + 3 \times 3^2 + 5 \times 4^2 + \ldots \text{ to } n \text{ terms}$$

QUES-02

$$3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots \text{ to } n \text{ terms}$$

QUES-03

$$\frac{1}{(1 \times 2)} + \frac{1}{(2 \times 3)} + \frac{1}{(3 \times 4)} + \ldots \text{ to } n \text{ terms}$$

QUES-04

$$\frac{1}{(1 \times 3)} + \frac{1}{(3 \times 5)} + \frac{1}{(5 \times 7)} + \ldots + \frac{1}{(2n-1)(2n+1)}$$

ANSWERS

ANS-01

$$\frac{n}{2}(n^3 + 4n^2 + 4n - 1)$$

ANS-02

$$\frac{n(n+1)(3n^2 + 5n + 1)}{6}$$

ANS-03

$$\frac{n}{(n+1)}$$

ANS-04

$$\frac{n}{(2n+1)}$$

- ***THANK YOU***

Lecture-15,16

Relation between A.M. and G.M

Relation between A.M and G.M:-

If A and G are respectively the arithmetic and geometric means between two distinct positive numbers a and b then

$$\boxed{A > G}$$

Q-1 Find two positive numbers whose difference is 12 and whose A.M exceed the G.M by 2. [2011-12]

Sol. Let the positive numbers be a & b

Given as $a-b=12 \longrightarrow (1)$

$$A.M \text{ of } a \,\&\, b = G.M \text{ of } a \,\&\, b + 2$$

$$\frac{a+b}{2} = \sqrt{ab} + 2$$

$$\frac{a+b}{2} - 2 = \sqrt{ab}$$

$$\frac{a+b-4}{2} = \sqrt{ab}$$

By (1)

$$\frac{a+b-4}{2} = \sqrt{ab}$$

By (1)

$a = 12 + b$

$$\frac{12+b+b-4}{2} = \sqrt{(12+b)\cdot b}$$

$$\frac{8+2b}{2} = \sqrt{12b+b^2}$$

$$4+b = \sqrt{12b+b^2}$$

$$(4+b)^2 = 12b+b^2$$

$$16 + \cancel{b^2} + 8b = 12b + \cancel{b^2}$$

$$\Rightarrow \quad 16 = 4b$$

$$\Rightarrow \quad b = 4$$

By ① $a = 12 + b = 12 + 4 = 16$ Ans

EXAMPLE *Find two positive numbers a and b whose AM and GM are 34 and 16 respectively.*

SOLUTION We have

$$\frac{a+b}{2} = 34 \text{ and } \sqrt{ab} = 16$$

$$\Rightarrow \quad a+b = 68 \text{ and } ab = 256$$

$$\Rightarrow \quad (a-b) = \sqrt{(a+b)^2 - 4ab} = \sqrt{(68)^2 - 4 \times 256} = \sqrt{3600} = \pm 60$$

$$\Rightarrow \quad a+b = 68 \text{ and } a-b = \pm 60$$

$$\Rightarrow \quad (a+b = 68, a-b = 60) \text{ or } (a+b = 68, a-b = -60)$$

$$\Rightarrow \quad (a = 64, b = 4) \text{ or } (a = 4, b = 64)$$

Hence, the required numbers are $(a = 64, b = 4)$ or $(a = 4, b = 64)$.

Some Special Series :-

(1) $(1+2+3+ \cdots + n) = \frac{n}{2}(n+1)$

$$\sum_{n=1}^{n} n = \frac{1}{2} n(n+1)$$

(2) $(1^2+2^2+3^2+ \cdots + n^2) = \frac{n}{6}(n+1)(2n+1)$

$$\sum_{n=1}^{n^2} n^2 = \frac{n}{6}(n+1)(2n+1)$$

(3) $(1^3+2^3+3^3+\cdots+n^3)=\left[\frac{n}{2}(n+1)\right]^2$

$$\sum_{n=1}^{n} n^3 = \left(\frac{n}{2}\right)^2 (n+1)^2$$

EXAMPLE 1 *If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that* $9S_2^2 = S_3(1+8S_1)$.

SOLUTION We have

$$S_1 = (1+2+3+\ldots+n) \Rightarrow S_1 = \frac{1}{2}n(n+1);$$

$$S_2 = (1^2+2^2+3^2+\ldots+n^2) \Rightarrow S_2 = \frac{1}{6}n(n+1)(2n+1);$$

and $$S_3 = (1^2+2^3+3^3+\ldots+n^3) \Rightarrow S_3 = \frac{1}{4}n^2(n+1)^2.$$

$$\therefore \quad 9S_2^2 = 9 \times \frac{1}{36} \cdot n^2(n+1)^2(2n+1)^2 = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

$$\text{And, } S_3(1+8S_1) = \frac{1}{4}n^2(n+1)^2 \cdot \left\{1 + 8 \cdot \frac{1}{2}n(n+1)\right\}$$

$$= \frac{1}{4}n^2(n+1)^2(4n^2+4n+1) = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$$

Hence, $9S_2^2 = S_3(1+8S_1)$.

EXAMPLE *Sum the series* $3 \cdot 8 + 6 \cdot 11 + 9 \cdot 14 + \ldots$ *to* n *terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 3, 6, 9, \ldots) \times (k\text{th term of } 8, 11, 14, \ldots)$$
$$= \{3 + (k-1) \times 3\} \times \{8 + (k-1) \times 3\} = 3k(3k+5)$$
$$= (9k^2 + 15k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} (9k^2 + 15k) = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right)$$
$$= 9 \cdot \left\{\frac{1}{6} n(n+1)(2n+1)\right\} + 15 \cdot \left\{\frac{1}{2} n(n+1)\right\}$$
$$= \frac{3}{2} n(n+1)\{(2n+1)+5\} = 3n(n+1)(n+3).$$

Hence, the required sum is $3n(n+1)(n+3)$.

EXAMPLE *Find the sum to n terms of the series whose nth term is* $(2n-1)^2$.

SOLUTION We have, $T_k = (2k-1)^2 = (4k^2 - 4k + 1)$.

$\therefore$ sum to n terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (4k^2 - 4k + 1) = 4 \cdot \sum_{k=1}^{n} k^2 - 4 \cdot \sum_{k=1}^{n} k + n$$

$[\because\ 1 + 1 + \ldots n \text{ times} = n]$

$$= 4 \cdot \frac{1}{6} n(n+1)(2n+1) - 4 \cdot \frac{1}{2} n(n+1) + n$$

$$= \frac{2}{3} n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{1}{3} \cdot \{2n(n+1)(2n+1) - 6n(n+1) + 3n\}$$

$$= \frac{1}{3}[n(n+1)\{2(2n+1) - 6\} + 3n] = \frac{1}{3} \cdot [4n(n+1)(n-1) + 3n]$$

$$= \frac{1}{3}(4n^3 - n) = \frac{1}{3} n(4n^2 - 1).$$

Hence, the required sum is $\frac{1}{3} n(4n^2 - 1)$.

EXAMPLE Sum the series $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ to n terms.

SOLUTION We have

$$T_k = (k\text{th term of } 1, 2, 3, \ldots) \times (k\text{th term of } 2, 3, 4, \ldots) \times (k\text{th term of } 3, 4, 5, \ldots)$$

$$= \{1 + (k-1) \times 1\} + \{2 + (k-1) \times 1\} \times \{3 + (k-1) \times 1\}$$

$$= k(k+1)(k+2) = (k^3 + 3k^2 + 2k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$\therefore \qquad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k) = \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$

$$= \frac{1}{4} n^2 (n+1)^2 + 3 \cdot \frac{1}{6} n(n+1)(2n+1) + 2 \cdot \frac{1}{2} n(n+1)$$

$$\left\{ \because \sum_{k=1}^{n} k^3 = \left\{\frac{1}{2} n(n+1)\right\}^2, \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1), \right.$$

$$\left. \sum_{k=1}^{n} k = \frac{1}{2} n(n+1) \right\}$$

$$= \frac{1}{4} n(n+1)\{n(n+1) + 2(2n+1) + 4\}$$

$$=\frac{1}{4}n(n+1)(n^2+5n+6)=\frac{1}{4}n(n+1)(n+2)(n+3).$$

Hence, the required sum is $\frac{1}{4}n(n+1)(n+2)(n+3)$.

EXAMPLE *Find the sum of the series $1^2+3^2+5^2+\dots$ to n terms.*

SOLUTION We have

$$T_k = k\text{th term of } (1^2+3^2+5^2+\dots)$$
$$= \{1+(k-1)\times 2\}^2 = (2k-1)^2 = (4k^2-4k+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}(4k^2-4k+1)$$
$$= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + (1+1+\dots n \text{ times})$$
$$= 4\cdot\frac{1}{6}n(n+1)(2n+1) - 4\cdot\frac{1}{2}n(n+1)+n$$
$$= \frac{n}{3}\cdot\{2(n+1)(2n+1)-6(n+1)+3\} = \frac{n}{3}(4n^2-1).$$

Hence, the required sum is $\frac{n}{3}(4n^2-1)$.

EXAMPLE

Find the sum of n terms of the series

$$\frac{1}{(2\times 5)}+\frac{1}{(5\times 8)}+\frac{1}{(8\times 11)}+\cdots.$$

SOLUTION We have

$$T_k = \frac{1}{(k\text{th term of } 2, 5, 8, \ldots) \times (k\text{th term of } 5, 8, 11, \ldots)}$$

$$= \frac{1}{\{2 + (k-1) \times 3\} \times \{5 + (k-1) \times 3\}}$$

$$= \frac{1}{(3k-1)(3k+2)} = \frac{1}{3}\left\{\frac{1}{(3k-1)} - \frac{1}{(3k+2)}\right\}.$$

$$\therefore \quad T_k = \frac{1}{3}\left\{\frac{1}{(3k-1)} - \frac{1}{(3k+2)}\right\}. \qquad \ldots \text{(i)}$$

Putting $k = 1, 2, 3, \ldots, n$ successively in (i), we get

$$T_1 = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$T_2 = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right)$$

… … … …

… … … …

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)} - \frac{1}{(3n+2)}\right\}.$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)} - \frac{1}{(3n+2)}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \ldots + T_n)$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{n}{2(3n+2)}.$$

PRACTICE QUESTIONS

QUES-01

$$1 \times 2^2 + 3 \times 3^2 + 5 \times 4^2 + \ldots \text{ to } n \text{ terms}$$

QUES-02

$$3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots \text{ to } n \text{ terms}$$

QUES-03

$$\frac{1}{(1 \times 2)} + \frac{1}{(2 \times 3)} + \frac{1}{(3 \times 4)} + \ldots \text{ to } n \text{ terms}$$

QUES-04

$$\frac{1}{(1 \times 3)} + \frac{1}{(3 \times 5)} + \frac{1}{(5 \times 7)} + \ldots + \frac{1}{(2n-1)(2n+1)}$$

ANSWERS

ANS-01

$$\frac{n}{2}(n^3 + 4n^2 + 4n - 1)$$

ANS-02

$$\frac{n(n+1)(3n^2 + 5n + 1)}{6}$$

ANS-03

$$\frac{n}{(n+1)}$$

ANS-04

$$\frac{n}{(2n+1)}$$

- ***THANK YOU***