UNIT-4

Lecture- 29

Introduction of Limit

LIMIT *We say that* $\lim_{x \to a} f(x) = l$ *if whenever* $x \to a, f(x) \to l$.

Working rules for finding $\lim_{x \to a} f(x)$

RULE I Put $x = a$ in the given function. If $f(a)$ is a definite value then

$$\lim_{x \to a} f(x) = f(a).$$

EXAMPLES 1. $\lim_{x \to 0} \sin x = \sin 0 = 0.$

2. $\lim_{x \to 1} (x^2 + 5x - 2) = (1^2 + 5 \times 1 - 2) = 4.$

RULE II If $f(x)$ is a rational function then factorize the numerator and the denominator. Cancel out the common factors and then put $x = a$.

EXAMPLE *Evaluate* (*i*) $\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3} \right)$ (*ii*) $\lim_{x \to 1} \frac{(x^2 - 4x + 3)}{(x - 1)}$.

SOLUTION (i) $\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3} \right) = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x \to 3} (x+3) = 6.$

(ii) $\lim_{x \to 1} \frac{(x^2 - 4x + 3)}{(x - 1)} = \lim_{x \to 1} \frac{(x-1)(x-3)}{(x-1)} = \lim_{x \to 1} (x-3) = -2.$

EXAMPLE *Evaluate* $\lim_{x \to 0} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \right\}$.

SOLUTION We have

$$\lim_{x\to 0}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right\}=\lim_{x\to 0}\left\{\frac{(\sqrt{1+x}-\sqrt{1-x}}{x}\cdot\frac{(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1-x})}\right\}$$

$$= \lim_{x \to 0} \frac{\{(1+x) - (1-x)\}}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2}{(\sqrt{1+x} + \sqrt{1-x})} = 1 \qquad [\text{putting } x = 0].$$

Fundamental Theorems on Limits (without proof)

(i) $\lim_{x \to a} \{f(x) + g(x)\} = \left\{\lim_{x \to a} f(x)\right\} + \left\{\lim_{x \to a} g(x)\right\}$

(ii) $\lim_{x \to a} \{f(x) - g(x)\} = \left\{\lim_{x \to a} f(x)\right\} - \left\{\lim_{x \to a} g(x)\right\}$

(iii) $\lim_{x \to a} \{c \cdot f(x)\} = c \cdot \left\{\lim_{x \to a} f(x)\right\}$, where c is a constant

(iv) $\lim_{x \to a} \{f(x) \cdot g(x)\} = \left\{\lim_{x \to a} f(x)\right\} \cdot \left\{\lim_{x \to a} g(x)\right\}$

(v) $\lim_{x \to a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\left\{ \lim_{x \to a} f(x) \right\}}{\left\{ \lim_{x \to a} g(x) \right\}}$, provided $\lim_{x \to a} g(x) \neq 0$

SUMMARY

1. $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1}$, where $a > 0$.

2. $\lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) = 1$.

3. $\lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log_e a$.

4. $\lim_{x \to 0} (1 + x)^{1/x} = e$.

5. $\lim_{x \to 0} \frac{\log (1 + x)}{x} = 1$.

EXAMPLE 1 *Evaluate* $\lim_{x \to a} \left\{ \frac{x^{12} - a^{12}}{x - a} \right\}.$

EXAMPLE . *Evaluate* $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$.

SOLUTION $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

$$= \lim_{(x+2) \to (a+2)} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{(x+2) - (a+2)}$$

$$= \frac{3}{2} \cdot (a+2)^{\left(\frac{3}{2} - 1\right)} = \frac{3}{2}(a+2)^{1/2} \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}\right].$$

Evaluate:

$$(i)\ \lim_{x\to 0}\left(\frac{e^{-x}-1}{x}\right)\quad (ii)\ \lim_{x\to 0}\left(\frac{e^{x}-e^{-x}}{x}\right)\quad (iii)\ \lim_{x\to 0}\left(\frac{e^{x}+e^{-x}-2}{x^{2}}\right)$$

EXAMPLE : *Evaluate* $\lim_{x \to 0} \left(\frac{3^x - 2^x}{x} \right)$.

SOLUTION $\lim_{x \to 0} \left(\frac{3^x - 2^x}{x} \right) = \lim_{x \to 0} \left\{ \frac{(3^x - 1) - (2^x - 1)}{x} \right\}$

$$= \lim_{x \to 0} \left(\frac{3^x - 1}{x} \right) - \lim_{x \to 0} \left(\frac{2^x - 1}{x} \right)$$

$$= (\log 3 - \log 2) = \log \frac{3}{2} \qquad \left[\because \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a \right]$$

TRIGONOMETRIC LIMITS

$$(i) \quad \lim_{\theta \to 0} \sin\theta = 0 \qquad (ii) \quad \lim_{\theta \to 0} \cos\theta = 1$$

THEOREM 2 (i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

EXAMPLE 1 *Evaluate:*

(i) $\lim_{x \to 0} \frac{\sin 2x}{x}$ (ii) $\lim_{x \to 0} \frac{\sin 3x}{5x}$ (iii) $\lim_{x \to 0} \left(\frac{\sin ax}{\sin bx}\right)$

EXAMPLE 2 *Evaluate* $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$.

EXAMPLE *Evaluate:*

(i) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ (ii) $\lim_{x \to 0} \frac{(\operatorname{cosec} x - \cot x)}{x}$

SOLUTION (i) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x) \cos x}{\cos^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x) \cos x}{(1 - \sin^2 x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)} = 0 \qquad \left[\text{putting } x = \frac{\pi}{2}\right].$$

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- **THANK YOU**

Lecture-30

Continuity

Continuity

CONTINUITY AT A POINT *A real function $f(x)$ is said to be continuous at a point a of its domain if $\lim_{x \to a} f(x)$ exists and equals $f(a)$.*

Thus, $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a).$$

If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.

If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.

REMARK $f(x)$ is discontinuous at $x = a$ in each of the following cases:

(i) $f(a)$ is not defined

(ii) $\lim_{x \to a} f(x)$ does not exist

(iii) $\lim_{x \to a} f(x) \neq f(a)$

EXAMPLE 1 *Show that* $f(x) = x^3$ *is continuous at* $x = 2$.

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SOLUTION We have $f(2) = 2^3 = 8;$

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} (2+h)^3 = \lim_{h \to 0} (8 + h^3 + 12h + 6h^2) = 8;$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} (2-h)^3 = \lim_{h \to 0} (8 - h^3 - 12h + 6h^2) = 8.$$

$$\therefore \quad \lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = f(2).$$

Hence, $f(x)$ is continuous at $x = 2$.

EXAMPLE . *Discuss the continuity of the function* $f(x)$ *at* $x = 0$, *if*

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0. \end{cases}$$

SOLUTION Clearly, $f(0) = (2 \times 0 + 1) = 1.$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} [2(0+h)+1] = \lim_{h \to 0} (2h+1) = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} [2(0-h)-1)] = \lim_{h \to 0} (-2h-1) = -1.$$

Thus, $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE. *Show that the function* $f(x)=\begin{cases}3x-2, & \text{when } x\le 0\\ x+1, & \text{when } x>0\end{cases}$

is discontinuous at $x=0$.

SOLUTION We have, $f(0)=(3\times 0-2)=-2$.

$$\lim_{x\to 0+} f(x)=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0}(h+1)=1.$$

$$\lim_{x\to 0-} f(x)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}[3(-h)-2]=\lim_{h\to 0}(-3h-2)=-2.$$

$\therefore \lim_{x\to 0+} f(x)\neq \lim_{x\to 0-} f(x)$ and therefore, $\lim_{x\to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x=0$.

EXAMPLE : *Show that the function* $f(x)=\begin{cases} 3x-2, & when \quad x \le 0 \\ x+1, & when \quad x > 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION We have, $f(0) = (3 \times 0 - 2) = -2.$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} (h+1) = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} [3(-h) - 2] = \lim_{h \to 0} (-3h - 2) = -2.$$

$\therefore \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE *Show that the function* $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

EXAMPLE *Show that the function* $f(x) = \begin{cases} \frac{x}{|x|}, & when\ x \neq 0 \\ 1, & when\ x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION It is being given that $f(0) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{h}{|h|} = \lim_{h \to 0} \frac{h}{h} = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{-h}{|h|} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

$$\therefore \quad \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x).$$

So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE *Examine the continuity of the function*

$$f(x) = \begin{cases} \dfrac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad at \quad x = 0.$$

SOLUTION We have $f(0) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{|\sin(0+h)|}{(0+h)} = \lim_{h \to 0} \frac{|\sin h|}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{|\sin(-h)|}{-h} = \lim_{h \to 0} \frac{|-\sin h|}{-h} = \lim_{h \to 0} \frac{\sin h}{-h} = -1.$$

$\therefore \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$. So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE *Find the value of k for which*

$$f(x)=\begin{cases} kx+5, & \text{when } x\le 2 \\ x-1, & \text{when } x>2 \end{cases}$$

is continuous at $x=2$.

SOLUTION We have, $f(2) = (k \times 2 + 5) = (2k + 5)$.

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} \{(2+h) - 1\} = \lim_{h \to 0} (1+h) = 1.$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} f(2-h)$$

$$= \lim_{h \to 0} \{k(2-h) + 5\} = \lim_{h \to 0} \{(2k+5) - kh\} = (2k+5).$$

Now, $\lim_{x \to 2} f(x)$ exists only when $2k+5 = 1$, i.e., when $k = -2$.

When $k = -2$, we have $\lim_{x \to 2} f(x) = f(2) = 1$.

Hence, $f(x)$ is continuous at $x = 2$ when $k = -2$.

EXAMPLE 1 *If the function* $f(x)=\begin{cases} 3ax+b, & \text{for } x>1 \\ 11, & \text{for } x=1 \\ 5ax-2b, & \text{for } x<1 \end{cases}$

is continuous at $x=1$, *find the values of a and b.*

SOLUTION We have, $f(1) = 11$.

$$\lim_{x \to 1+} f(x) = \lim_{h \to 0} f(1+h)$$

$$= \lim_{h \to 0} \{3a(1+h) + b\} = \lim_{h \to 0} \{(3a+b) + 3ah\}$$

$$= (3a+b).$$

$$\lim_{x \to 1-} f(x) = \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} \{5a(1-h) - 2b\} = \lim_{h \to 0} \{(5a-2b) - 5ah\}$$

$$= (5a-2b).$$

Since $f(x)$ is continuous at $x = 1$, we have

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = f(1).$$

$\therefore \quad 3a + b = 5a - 2b = 11.$

On solving $(3a + b = 11)$ and $(5a - 2b = 11)$, we get $a = 3, \ b = 2$.

Hence, $a = 3, \ b = 2$.

EXAMPLE - *Show that the function* $f(x)=\begin{cases}\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right), & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$

is discontinuous at $x=0$.

SOLUTION Clearly, $f(0) = 0$.

Now, $\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}\left(\frac{e^{1/h}-1}{e^{1/h}+1}\right)$

$$= \lim_{h\to 0} \frac{e^{1/h}\left(1-\frac{1}{e^{1/h}}\right)}{e^{1/h}\left(1+\frac{1}{e^{1/h}}\right)} = \lim_{h\to 0} \frac{\left(1-\frac{1}{e^{1/h}}\right)}{\left(1+\frac{1}{e^{1/h}}\right)} = 1.$$

And, $\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\left(\frac{e^{-1/h}-1}{e^{-1/h}+1}\right)$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{e^{1/h}}-1\right)}{\left(\frac{1}{e^{1/h}}+1\right)} = -1.$$

Thus, $\lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$, and therefore, $\lim_{x\to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

PRACTICE QUESTIONS

Prove That

$$f(x)=\begin{cases}\dfrac{x^2-x-6}{x-3}, & \text{when } x\neq 3;\\ 5, & \text{when } x=3\end{cases}$$ is continuous at $x=3$.

$$f(x)=\begin{cases}\dfrac{x^2-25}{x-5}, & \text{when } x\neq 5;\\ 10, & \text{when } x=5\end{cases}$$ is continuous at $x=5$.

$$f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & \text{when } x\neq 0;\\ 1, & \text{when } x=0\end{cases}$$ is discontinuous at $x=0$.

$$f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & \text{when } x\neq 0;\\ 1, & \text{when } x=0\end{cases}$$ is discontinuous at $x=0$.

$$f(x)=\begin{cases}2-x, & \text{when } x<2;\\ 2+x, & \text{when } x\geq 2\end{cases}$$ is discontinuous at $x=2$.

$$f(x)=\begin{cases}3-x, & \text{when } x\leq 0;\\ x^2, & \text{when } x>0\end{cases}$$ is discontinuous at $x=0$.

- THANK YOU

Lecture- 31

Differentiability

Differentiability

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) **Right-hand derivative** $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Rf'(a)$.

(ii) **Left-hand derivative** $\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Lf'(a)$.

DIFFERENTIABILITY *A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.*

EXAMPLE 1 Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.

EXAMPLE 1 *Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.*

SOLUTION $$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \left(\frac{1 + h^2 + 2h - 1}{h} \right) = \lim_{h \to 0} (h+2) = 2.$$

And, $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)^2 - (1)^2}{-h}$$

$$= \lim_{h \to 0} \left(\frac{1 + h^2 - 2h - 1}{-h} \right) = \lim_{h \to 0} (-h+2) = 2.$$

$\therefore \quad Rf'(1) = Lf'(1) = 2.$

This shows that $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$.

EXAMPLE : *Show that the function* $f(x)=\begin{cases}1+x, & if \quad x\le 2;\\ 5-x, & if \quad x>2\end{cases}$ *is not differentiable at* $x=2$.

SOLUTION $Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\left[\frac{5-(2+h)-3}{h}\right]$

$$= \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} (-1) = -1.$$

And, $Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0}\left[\frac{1+(2-h)-3}{-h}\right] = \lim_{h\to 0} \frac{-h}{-h} = \lim_{h\to 0} 1 = 1.$$

Thus, $Rf'(2) \neq Lf'(2)$.

Hence, $f(x)$ is not differentiable at $x = 2$.

THEOREM *Every differentiable function is continuous. But, every continuous function need not be differentiable.*

EXAMPL'.f *Show that* $f(x) = |x-2|$ *is continuous but not differentiable at* $x = 2.$

SOLUTION We have $f(2) = |2 - 2| = 0.$

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} |2 + h - 2| = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0.$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} |2 - h - 2| = \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

$$\therefore \lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = f(2) = 0.$$

So, $f(x)$ is continuous at $x = 2$.

But, $$Rf'(2) = \lim_{h\to0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to0} \frac{|2+h-2|-0}{h}$$

$$= \lim_{h\to0} \frac{|h|}{h} = \lim_{h\to0} \frac{h}{h} = 1.$$

And, $$Lf'(2) = \lim_{h\to0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to0} \frac{|2-h-2|-0}{-h}$$

$$= \lim_{h\to0} \frac{|-h|}{-h} = \lim_{h\to0} \frac{h}{-h} = -1.$$

Thus, $Rf'(2) \neq Lf'(2)$.

This shows that $f(x)$ is not differentiable at $x = 2$.

PRACTICE QUESTIONS

Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

Let $f(x) = \begin{cases} (2 - x), & \text{when } x \geq 1 \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$

Show that $f(x)$ is continuous but not differentiable at $x = 1$.

Show that $f(x) = [x]$ is neither continuous nor derivable at $x = 2$.

Show that the function $f(x) = \begin{cases} (1 - x), & \text{when } x < 1; \\ (x^2 - 1), & \text{when } x \geq 1 \end{cases}$

is continuous but not differentiable at $x = 1$.

Let $f(x) = \begin{cases} (2 + x), & \text{if } x \geq 0 \\ (2 - x), & \text{if } x < 0. \end{cases}$ Show that $f(x)$ is not derivable at $x = 0$.

THANK YOU

Lecture- 32 & 33

Derivative of sum,difference, and its based questions

SUMMARY

We may summarise the above results as given below:

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(\sin x) = \cos x$

(iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(viii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$

The derivative of a constant function is zero, i.e., $\frac{d}{dx}(c) = 0.$

EXAMPLE 1 *Find the derivative of*

(i) $8x^3$ (ii) $6\sqrt{x}$ (iii) $5e^x$ (iv) 9×2^x

EXAMPLE 2 *Find the derivative of* $(x^3 + e^x + 3^x + \cot x)$ *with respect to* x.

EXAMPLE 3 *Find the derivative of* $\left(9x^2 + \frac{3}{x} + 5\sin x\right)$ *with respect to x.*

SOLUTION We have $\frac{d}{dx}\left(9x^2 + \frac{3}{x} + 5\sin x\right)$

$$= 9 \cdot \frac{d}{dx}\left(x^2\right) + 3 \cdot \frac{d}{dx}\left(x^{-1}\right) + 5 \cdot \frac{d}{dx}\left(\sin x\right)$$

$$= 9 \times 2x + 3 \cdot (-1)\, x^{-2} + 5\cos x = 18x - \frac{3}{x^2} + 5\cos x.$$

EXAMPLE 1 *Differentiate the following functions with respect to x:*

$$\left(x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e\right)$$

SOLUTION $\frac{d}{dx}\left(x^2+\frac{4}{x^2}-\frac{2}{3}\tan x+6e\right)$

$$=\frac{d}{dx}(x^2)+4\cdot\frac{d}{dx}(x^{-2})-\frac{2}{3}\cdot\frac{d}{dx}(\tan x)+6\cdot\frac{d}{dx}(e)$$

$$=2x+4\cdot(-2)\,x^{-3}-\frac{2}{3}\sec^2 x+6\times 0 \qquad \left[\because \frac{d}{dx}(e)=0\right]$$

$$=2x-\frac{8}{x^3}-\frac{2}{3}\sec^2 x.$$

EXAMPLE 2 *Find the derivative of* $\left\{\dfrac{3}{\sqrt[3]{x}} - \dfrac{5}{\cos x} + \dfrac{6}{\sin x} - \dfrac{2\tan x}{\sec x} + 7\right\}$.

EXAMPLE 3 *Differentiate the following functions:*

(i) $(x^2 - 5x + 6)(x - 3)$ (ii) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (iii) $\frac{3x^2 + 2x + 5}{\sqrt{x}}$

EXAMPLE 4 *If* $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}} = \tan x.$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x.$$

EXAMPLE 5 *If* $y = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty\right)\cdot$ *show that* $\frac{dy}{dx} = y.$

SOLUTION We have, $y = e^x.$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x = y.$$

EXAMPLE 6 If $u = 3t^4 - 5t^3 + 2t^2 - 18t + 4$, find $\frac{du}{dt}$ at $t = 1$.

SOLUTION $\frac{du}{dt}=\frac{d}{dt}(3t^4-5t^3+2t^2-18t+4)$

$$=3\cdot\frac{d}{dt}(t^4)-5\cdot\frac{d}{dt}(t^3)+2\cdot\frac{d}{dt}(t^2)-18\cdot\frac{d}{dt}(t)+\frac{d}{dt}(4)$$

$$=3\times 4t^3-5\times 3t^2+2\times 2t-18\times 1+0$$

$$=12t^3-15t^2+4t-18.$$

$$\therefore \quad \left(\frac{du}{dt}\right)_{t=1}=(12\times 1^3-15\times 1^2+4\times 1-18)$$

$$=(12-15+4-18)=-17.$$

DERIVATIVE OF THE PRODUCT OF FUNCTIONS

THEOREM (Product rule) *If $f(x)$ and $g(x)$ are two differentiable functions then $f(x)\cdot g(x)$ is also differentiable, and*

$$\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}.$$

EXAMPLE 1 *Differentiate:* (i) xe^x (ii) $x^2e^x \sin x$

EXAMPLE 2 *Differentiate* $x^2 \tan x$.

PRACTICE QUESTIONS

Differentiate:

$$(x^2 + 3x + 1)\sin x$$

$$x^n \cot x$$

-

- THANK YOU

Lecture- 34,35

Derivative of Quotient of functions and Composite Functions

Derivative of the Quotient of two Functions

THEOREM (Quotient rule) *If $f(x)$ and $g(x)$ are two differentiable functions and* $g(x) \neq 0$ *then* $\frac{f(x)}{g(x)}$ *is also differentiable, and*

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot \frac{d}{dx}\{f(x)\} - f(x) \cdot \frac{d}{dx}\{g(x)\}}{[g(x)]^2}.$$

EXAMPLE 1 *Differentiate:*

(i) $\dfrac{e^x}{x}$ (ii) $\left(\dfrac{2x+3}{x^2-5}\right)$ (iii) $\dfrac{e^x}{(1+\sin x)}$

EXAMPLE 2 *Differentiate* $\left(\dfrac{x^2+5x-6}{4x^2-x+3}\right)$.

EXAMPLE . If $y = \left\{ \frac{1 - \tan x}{1 + \tan x} \right\}$, show that $\frac{dy}{dx} = \frac{-2}{(1 + \sin 2x)}$.

SOLUTION By the quotient rule, we have

$$\frac{dy}{dx}=\frac{(1+\tan x)\cdot\frac{d}{dx}(1-\tan x)-(1-\tan x)\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)(-\sec^2 x)-(1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$=\frac{-2\sec^2 x}{(1+\tan x)^2}=\frac{-2}{(\cos^2 x)(1+\tan^2 x+2\tan x)}$$

$$=\frac{-2}{(\cos^2 x)\left\{1+\frac{\sin^2 x}{\cos^2 x}+\frac{2\sin x}{\cos x}\right\}}=\frac{-2}{(1+\sin 2x)}.$$

EXAMPLE 5 *Differentiate:*

(i) $\left(\dfrac{\sin x + \cos x}{\sin x - \cos x}\right)$ (ii) $\left(\dfrac{\sec x - 1}{\sec x + 1}\right)$

SOLUTION (i) $\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$

$$=\frac{(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-[(\sin x-\cos x)^2+(\sin x+\cos x)^2]}{(\sin x-\cos x)^2}$$

$$=\frac{-2(\sin^2 x+\cos^2 x)}{(\sin^2 x+\cos^2 x-2\sin x\cos x)}=\frac{-2}{(1-\sin 2x)}.$$

(ii) $$\frac{d}{dx}\left(\frac{\sec x - 1}{\sec x + 1}\right)$$

$$= \frac{(\sec x + 1) \cdot \frac{d}{dx}(\sec x - 1) - (\sec x - 1) \cdot \frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$$

$$= \frac{(\sec x + 1) \sec x \tan x - (\sec x - 1) \sec x \tan x}{(\sec x + 1)^2} = \frac{2\sec x \ \tan x}{(\sec x + 1)^2}.$$

PRACTICE QUESTIONS

Differentiate:

1. $\dfrac{2^x}{x}$

2. $\dfrac{\log x}{x}$

3. $\dfrac{e^x}{(1+x)}$

4. $\dfrac{e^x}{(1+x^2)}$

5. $\left(\dfrac{2x^2-4}{3x^2+7}\right)$

6. $\left(\dfrac{x^2+3x-1}{x+2}\right)$

7. $\dfrac{(x^2-1)}{(x^2+7x+1)}$

8. $\left(\dfrac{5x^2+6x+7}{2x^2+3x+4}\right)$

9. $\dfrac{x}{(a^2+x^2)}$

10. $\dfrac{x^4}{\sin x}$

11. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

12. $\dfrac{\cos x}{\log x}$

ANSWERS

1. $\dfrac{2^x(x\log 2-1)}{x^2}$ 2. $\dfrac{(1-\log x)}{x^2}$ 3. $\dfrac{xe^x}{(1+x)^2}$ 4. $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$

5. $\dfrac{52x}{(3x^2+7)^2}$ 6. $\dfrac{(x^2+4x+7)}{(x+2)^2}$ 7. $\dfrac{(7x^2+4x+7)}{(x^2+7x+1)^2}$

8. $\dfrac{3(x^2+4x+1)}{(2x^2+3x+4)^2}$ 9. $\dfrac{(a^2-x^2)}{(a^2+x^2)^2}$ 10. $\dfrac{x^3(4\sin x-x\cos x)}{\sin^2 x}$

ANSWERS

11. $\dfrac{\sqrt{a}}{\sqrt{x}\cdot(\sqrt{a}-\sqrt{x})^2}$

12. $\dfrac{-(x\sin x\log x+\cos x)}{x(\log x)^2}$

Derivative of Quotient of functions and Composite Functions

Derivative of a Function of a Function

CHAIN RULE If $y = f(t)$ and $t = g(x)$ then $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$.

This rule may be extended further.

If $y = f(t), t = g(u)$ and $u = h(x)$ then $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}\right)$.

EXAMPLE 1 *Differentiate* (i) $\sin x^3$ (ii) $\sin^3 x$ (iii) $e^{\sin x}$

EXAMPLE 2 $If\ y = \dfrac{1}{\sqrt{a^2 - x^2}}$, $find\ \dfrac{dy}{dx}$.

SOLUTION Put $(a^2 - x^2) = t$, so that $y = \frac{1}{\sqrt{t}} = t^{-1/2}$ and $t = (a^2 - x^2)$.

$$\therefore \qquad \frac{dy}{dt} = -\frac{1}{2}t^{-3/2} \text{ and } \frac{dt}{dx} = -2x.$$

So, $$\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(-\frac{1}{2}t^{-3/2}\right)(-2x) = xt^{-3/2} = x(a^2 - x^2)^{-3/2}.$$

EXAMPLE 3 *Differentiate:*

(i) $(ax + b)^m$ (ii) $(3x + 5)^6$

EXAMPLE *Differentiate* $e^{\sqrt{\cot x}}$.

SOLUTION Let $y = e^{\sqrt{\cot x}}$. Put $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, so that

$$y = e^u, \; u = \sqrt{t} \text{ and } t = \cot x.$$

$$\therefore \quad \frac{dy}{du} = e^u, \; \frac{du}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = -\operatorname{cosec}^2 x.$$

$$\text{So, } \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = -\frac{1}{2} \cdot \frac{\operatorname{cosec}^2 x}{\sqrt{t}} e^u$$

$$= \frac{-\operatorname{cosec}^2 x}{2\sqrt{t}} \cdot e^{\sqrt{t}} \quad [\because u = \sqrt{t}]$$

$$= \frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}} \cdot e^{\sqrt{\cot x}} \quad [\because t = \cot x].$$

EXAMPLE $If\ y = \cos^2 x^2,\ find\ \frac{dy}{dx}.$

SOLUTION $y = (\cos x^2)^2$. Put $x^2 = t$ and $\cos x^2 = \cos t = u$, so that

$$y = u^2,\ u = \cos t \text{ and } t = x^2.$$

$$\therefore \quad \frac{dy}{du} = 2u,\ \frac{du}{dt} = -\sin t \text{ and } \frac{dt}{dx} = 2x.$$

$$\text{So, } \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= -4ux \sin t = -4x \sin t \cos t \qquad [\because\ u = \cos t]$$

$$= -4x \sin x^2 \cos x^2 = -2x \sin(2x^2) \qquad [\because\ t = x^2].$$

EXAMPLE : *Differentiate* $\sqrt{\frac{1-\tan x}{1+\tan x}}$.

EXAMPLE If $y = \sin(\sqrt{\sin x + \cos x})$, find $\frac{dy}{dx}$.

SOLUTION Putting $(\sin x + \cos x) = t$ and $\sqrt{(\sin x + \cos x)} = \sqrt{t} = u$, we get

$$y = \sin u,\; u = \sqrt{t} \text{ and } t = (\sin x + \cos x).$$

$$\therefore \quad \frac{dy}{du} = \cos u,\; \frac{du}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$$

and $\frac{dt}{dx} = (\cos x - \sin x)$.

So,
$$\frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = \frac{\cos u}{2\sqrt{t}} \cdot (\cos x - \sin x)$$

$$= \frac{\cos\sqrt{t}}{2\sqrt{t}} \cdot (\cos x - \sin x) \qquad [\because\; u = \sqrt{t}]$$

$$= \frac{\cos(\sqrt{\sin x + \cos x})(\cos x - \sin x)}{2\sqrt{\sin x + \cos x}} \quad [\because\; t = (\sin x + \cos x)].$$

EXAMPLE : *Differentiate* $e^{ax}\cos(bx+c)$.

SOLUTION Using the product rule, we have

$$\frac{d}{dx}\{e^{ax}\cos(bx+c)\}$$

$$= e^{ax}\cdot\frac{d}{dx}\{\cos(bx+c)\} + \cos(bx+c)\cdot\frac{d}{dx}(e^{ax})$$

$$= e^{ax}\{-\sin(bx+c)\}\cdot\frac{d}{dx}(bx+c) + \cos(bx+c)\cdot e^{ax}\cdot\frac{d}{dx}(ax)$$

[using the chain rule]

$$= -be^{ax}\sin(bx+c) + ae^{ax}\cos(bx+c)$$

$$= e^{ax}[a\cos(bx+c) - b\sin(bx+c)].$$

PRACTICE QUESTIONS

1. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, show that $\frac{dy}{dx} + y^2 + 1 = 0$.

2. If $y = \frac{\cos x + \sin x}{\cos x - \sin x}$, show that $\frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$.

3. If $y = \sqrt{\frac{1 - x}{1 + x}}$, prove that $(1 - x^2)\frac{dy}{dx} + y = 0$.

4. If $y = \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}}$, show that $\frac{dy}{dx} = \sec x\,(\tan x + \sec x)$.

THANK YOU

Lecture- 36

Derivatives of Inverse Trignometric Function

SUMMARY

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}$

(iv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\cdot\sqrt{x^2-1}}$

EXAMPLE 1 *Differentiate the following w.r.t. x:*

(i) $\sin^{-1}2x$ (ii) $\tan^{-1}\sqrt{x}$ (iii) $\cos^{-1}(\cot x)$

SOLUTION

(i) Let $y = \sin^{-1} 2x$.

Putting $2x = t$, we get $y = \sin^{-1} t$ and $t = 2x$.

Now, $y = \sin^{-1} t \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$.

And, $t = 2x \Rightarrow \frac{dt}{dx} = 2$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{2}{\sqrt{1-t^2}} = \frac{2}{\sqrt{1-4x^2}} \qquad [\because \quad t = 2x].$$

Hence, $\frac{d}{dx}(\sin^{-1} 2x) = \frac{2}{\sqrt{1-4x^2}}$.

(ii) Let $y = \tan^{-1} \sqrt{x}$.

Putting $\sqrt{x} = t$, we get $y = \tan^{-1} t$ and $t = \sqrt{x}$.

Now, $y = \tan^{-1} t \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{(1+t^2)}$.

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{1}{(1+t^2)} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}$ $\quad [\because t = \sqrt{x}]$.

Hence, $\dfrac{d}{dx}(\tan^{-1}\sqrt{x}) = \dfrac{1}{2\sqrt{x}(1+x)}$.

(iii) Let $y = \cos^{-1}(\cot x)$.

Putting $\cot x = t$, we get $y = \cos^{-1} t$ and $t = \cot x$.

Now, $y = \cos^{-1} t \Rightarrow \dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}}$.

And, $t = \cot x \Rightarrow \dfrac{dt}{dx} = -\operatorname{cosec}^2 x$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{\operatorname{cosec}^2 x}{\sqrt{1-t^2}} = \frac{\operatorname{cosec}^2 x}{\sqrt{1-\cot^2 x}} \qquad [\because\ t = \cot x].$$

Hence, $\dfrac{d}{dx}\{\cos^{-1}(\cot x)\} = \dfrac{\operatorname{cosec}^2 x}{\sqrt{1-\cot^2 x}}$.

EXAMPLE 2 *Differentiate the following w.r.t. x:*

(i) $\sec(\tan^{-1}x)$ *(ii)* $\sin(\tan^{-1}x)$ *(iii)* $\cot(\cos^{-1}x)$

SOLUTION (i) Let $y = \sec(\tan^{-1}x)$.

Putting $\tan^{-1}x = t$, we get $y = \sec t$ and $t = \tan^{-1}x$.

Now, $y = \sec t \Rightarrow \frac{dy}{dt} = \sec t \tan t$.

And, $t = \tan^{-1}x \Rightarrow \frac{dt}{dx} = \frac{1}{(1+x^2)}$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{\sec t \tan t}{(1+x^2)} = \frac{(\sqrt{1+\tan^2 t})(\tan t)}{(1+x^2)}$$

$$= \frac{(\sqrt{1+x^2})x}{(1+x^2)} = \frac{x}{\sqrt{1+x^2}} \quad [\because t = \tan^{-1}x \Rightarrow \tan t = x]$$

Hence, $\frac{d}{dx}\{\sec(\tan^{-1}x)\} = \frac{x}{\sqrt{1+x^2}}$.

(ii) Let $y = \sin(\tan^{-1} x)$.

Putting $\tan^{-1} x = t$, we get $y = \sin t$ and $t = \tan^{-1} x$.

Now, $y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t$.

And, $t = \tan^{-1} x \;\Rightarrow\; \dfrac{dt}{dx} = \dfrac{1}{(1+x^2)}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \cos t \cdot \frac{1}{(1+x^2)} = \frac{1}{(1+x^2)^{3/2}}$$

$$\left[\because \;\; \tan t = x \;\Rightarrow\; \cos t = \frac{1}{\sqrt{1+x^2}}\right].$$

Hence, $\dfrac{d}{dx}\{\sin(\tan^{-1} x)\} = \dfrac{1}{(1+x^2)^{3/2}}$.

(iii) Let $y = \cot(\cos^{-1} x)$.

Putting $\cos^{-1} x = t$, we get $y = \cot t$ and $t = \cos^{-1} x$.

Now, $y = \cot t \;\Rightarrow\; \dfrac{dy}{dt} = -\operatorname{cosec}^2 t$.

And, $t = \cos^{-1} x \;\Rightarrow\; \dfrac{dt}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{\operatorname{cosec}^2 t}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{3/2}}$$

$$\left[\because \;\; \cos t = x \;\Rightarrow\; \operatorname{cosec}^2 t = \frac{1}{(1-x^2)}\right].$$

Hence, $\dfrac{d}{dx}\{\cot(\cos^{-1} x)\} = \dfrac{1}{(1-x^2)^{3/2}}$.

EXAMPLE 4 *Differentiate* $\sqrt{\cot^{-1}\sqrt{x}}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{\cot^{-1}\sqrt{x}}$.

Putting $\sqrt{x} = t$ and $\cot^{-1}\sqrt{x} = \cot^{-1}t = u$, we get

$$y = \sqrt{u}, \text{ where } u = \cot^{-1}t \text{ and } t = \sqrt{x}.$$

Now, $y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$;

$$u = \cot^{-1}t \Rightarrow \frac{du}{dt} = \frac{-1}{(1+t^2)}.$$

And, $t = \sqrt{x} \Rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = \frac{-1}{4\sqrt{u}(1+t^2)\sqrt{x}}$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}t})(1+t^2)\sqrt{x}} \qquad [\because \ u = \cot^{-1}t]$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}\sqrt{x}})(1+x)\sqrt{x}} \qquad [\because \ t = \sqrt{x}].$$

EXAMPLE 5 *Differentiate* $e^{\tan^{-1}\sqrt{x}}$ *w.r.t. x.*

SOLUTION Let $y = e^{\tan^{-1}\sqrt{x}}$.

Putting $\sqrt{x} = t$ and $\tan^{-1}\sqrt{x} = \tan^{-1} t = u$, we get

$$y = e^u, \text{ where } u = \tan^{-1} t \text{ and } t = \sqrt{x}.$$

Now, $y = e^u \Rightarrow \dfrac{dy}{du} = e^u$;

$$u = \tan^{-1} t \Rightarrow \frac{du}{dt} = \frac{1}{(1+t^2)}.$$

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = e^u \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}}$$

$$= e^{\tan^{-1} t} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \qquad [\because \ u = \tan^{-1} t]$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = e^{u} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}}$$

$$= e^{\tan^{-1} t} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \qquad [\because \quad u = \tan^{-1} t]$$

$$= \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)} \qquad [\because \quad t = \sqrt{x}].$$

Hence, $\dfrac{dy}{dx} = \dfrac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$.

EXAMPLE ' *Show that* $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]=\sqrt{a^2-x^2}.$

SOLUTION We have

$$\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}\right]+\frac{a^2}{2}\cdot\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{x}{2}\cdot\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)+\left(\sqrt{a^2-x^2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$=\frac{x}{2}\cdot\frac{1}{2}(a^2-x^2)^{-1/2}\cdot(-2x)+\left(\sqrt{a^2-x^2}\right)\cdot\frac{1}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{x}{2} \cdot \frac{d}{dx}(\sqrt{a^2 - x^2}) + (\sqrt{a^2 - x^2}) \cdot \frac{d}{dx}\left(\frac{x}{2}\right) + \frac{a^2}{2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{x}{2} \cdot \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot (-2x) + (\sqrt{a^2 - x^2}) \cdot \frac{1}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{-x^2 + (a^2 - x^2) + a^2}{2\sqrt{a^2 - x^2}} = \frac{(a^2 - x^2)}{\sqrt{(a^2 - x^2)}} = \sqrt{(a^2 - x^2)}.$$

Hence, $\frac{d}{dx}\left[\frac{x}{2} \cdot \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \sin^{-1}\frac{x}{a}\right] = \sqrt{a^2 - x^2}.$

PRACTICE QUESTIONS

Differentiate each of the following w.r.t. x:

1. $\cos^{-1}2x$
2. $\tan^{-1}x^2$
3. $\sec^{-1}\sqrt{x}$
4. $\sin^{-1}\frac{x}{a}$
5. $\tan^{-1}(\log x)$
6. $\cot^{-1}(e^x)$
7. $\log(\tan^{-1}x)$
8. $\cot^{-1}x^3$
9. $\sin^{-1}(\cos x)$
10. $(1+x^2)\tan^{-1}x$
11. $\tan^{-1}(\cot x)$

ANSWERS

1. $\dfrac{-2}{\sqrt{1-4x^2}}$ 2. $\dfrac{2x}{(1+x^4)}$ 3. $\dfrac{1}{2x\sqrt{x-1}}$ 4. $\dfrac{1}{\sqrt{a^2-x^2}}$

5. $\dfrac{1}{x\{1+(\log x)^2\}}$ 6. $\dfrac{-e^x}{(1+e^{2x})}$ 7. $\dfrac{1}{(1+x^2)\tan^{-1}x}$ 8. $\dfrac{-3x^2}{(1+x^6)}$

9. -1 10. $(1+2x\tan^{-1}x)$ 11. -1

- ***<u>THANK YOU</u>***

Lecture- 37

Derivatives of Exponential, Logarithm and Parametric Form

Derivatives of Exponential and Logarithmic Functions

We have (i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(\log x) = \frac{1}{x}\cdot$

$$\text{(ii)}\ \frac{d}{dx}(a^x) = a^x(\log a)$$

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) e^{x^2} (ii) e^{-3x} (iii) $e^{\cos x}$

..2

EXAMPLE 2 *Differentiate each of the following w.r.t. x:*

(i) $\sin(\log x), \quad x > 0$ *(ii)* $\log(\log x), \quad x > 1$

SOLUTION (i) Let $y = \sin(\log x)$.

Putting $\log x = t$, we get

$$y = \sin t \text{ and } t = \log x$$

$$\Rightarrow \frac{dy}{dt} = \cos t \text{ and } \frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \left(\cos t \times \frac{1}{x}\right) = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}.$$

Hence, $\frac{d}{dx}\{\sin(\log x)\} = \frac{\cos(\log x)}{x}$.

(ii) Let $y = \log(\log x)$.

Putting $\log x = t$, we get

$$y = \log t \text{ and } t = \log x$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{t} \text{ and } \frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \left(\frac{1}{t} \times \frac{1}{x}\right) = \left(\frac{1}{\log x} \times \frac{1}{x}\right) = \frac{1}{(x \log x)}.$$

$$\therefore \frac{d}{dx}\{\log(\log x)\} = \frac{1}{(x \log x)}.$$

M.A.

EXAMPLE 3 $If\ y = e^{\sqrt{\cot x}}, find\ \frac{dy}{dx}.$

SOLUTION Given: $y = e^{\sqrt{\cot x}}$.

Putting $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, we get

$$y = e^u, \quad u = \sqrt{t} \text{ and } t = \cot x$$

$$\Rightarrow \frac{dy}{du} = e^u, \quad \frac{du}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = -\text{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right)$$

$$= \left\{ e^u \cdot \frac{1}{2\sqrt{t}} \cdot (-\text{cosec}^2 x) \right\} = e^{\sqrt{\cot x}} \cdot \frac{1}{2\sqrt{\cot x}} \cdot (-\text{cosec}^2 x)$$

$$= \frac{(-\text{cosec}^2 x)\, e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}.$$

EXAMPLE 4 $If\ y = \log\ \tan \frac{x}{2}, find\ \frac{dy}{dx}$.

SOLUTION Given: $y = \log \tan \frac{x}{2}$.

Putting $\frac{x}{2} = t$ and $\tan \frac{x}{2} = \tan t = u$, we get

$$y = \log u, \quad u = \tan t \text{ and } t = \frac{x}{2}$$

$$\Rightarrow \quad \frac{dy}{du} = \frac{1}{u}, \quad \frac{du}{dt} = \sec^2 t \text{ and } \frac{dt}{dx} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right)$$

PRACTICE QUESTIONS

EXERCISE 10B

Differentiate each of the following w.r.t. x:

1. (i) e^{4x} (ii) e^{-5x} (iii) e^{x^3}

2. (i) $e^{2/x}$ (ii) $e^{\sqrt{x}}$ (iii) $e^{-2\sqrt{x}}$

3. (i) $e^{\cot x}$ (ii) $e^{-\sin 2x}$ (iii) $e^{\sqrt{\sin x}}$

4. (i) $\tan(\log x)$ (ii) $\log \sec x$ (iii) $\log \sin \frac{x}{2}$

5. (i) $\log_3 x$ (ii) 2^{-x} (iii) 3^{x+2}

6. (i) $\log\left(x+\frac{1}{x}\right)$ (ii) $\log \sin 3x$ (iii) $\log\left(x+\sqrt{1+x^2}\right)$

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) $\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$ (ii) $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)$

SOLUTION (i) Let $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) = \tan^{-1}\left\{\frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\}$

$$= \tan^{-1}\left\{\tan\frac{x}{2}\right\} = \frac{x}{2}.$$

$\therefore \quad y = \frac{x}{2}.$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}.$

(ii) Let $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$

[on dividing num. and denom. by cos x]

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - x\right)\right\} = \left(\frac{\pi}{4} - x\right).$

$\therefore \quad y = \left(\frac{\pi}{4} - x\right).$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}(x) = (0 - 1) = -1.$

EXAMPLE 2 *Differentiate w.r.t. x:*

(i) $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$

(ii) $\tan^{-1}(\sec x + \tan x)$

SOLUTION (i) Let $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right\}$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}-\frac{x}{2}\right).$$

$$\therefore \quad y = \left(\frac{\pi}{4}-\frac{x}{2}\right).$$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0-\frac{1}{2}\right) = \frac{-1}{2}.$

(ii) Let $y = \tan^{-1}(\sec x + \tan x)$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left\{\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right\}$$

$$\left\{\because \cos\left(\frac{\pi}{2}+x\right) = -\sin x;\ \sin\left(\frac{\pi}{2}+x\right) = \cos x\right\}$$

EXAMPLE 12 *Differentiate* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ *w.r.t. x.*

SOLUTION Let $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$.

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\} = \tan^{-1}\left\{\frac{\sec\theta - 1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\frac{1}{\cos\theta}-1\right)}{\sin\theta}\cdot\cos\theta\right\} = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\} = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad y = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}(\tan^{-1}x) = \frac{1}{2(1+x^2)}\cdot$$

PRACTICE QUESTIONS

Differentiate w.r.t. x:

$$(i)\ \tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\} \quad (ii)\ \tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}$$

$$If\ y = \tan^{-1}\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})},\ find\ \frac{dy}{dx}.$$

PRACTICE QUESTIONS

Differentiate w.r.t. x:

(i) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (ii) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (iii) $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

- ***THANK YOU***

Lecture- 38

Logarithmic Differentiation & Derivative of implicit Function

EXAMPLE 9 *Differentiate* x^x *w.r.t.* x.

SOLUTION Let $y = x^x$.

Taking logarithm on both sides of (i), we get

$$\log y = x \log x.$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$$

$$= \left(x \cdot \frac{1}{x} + \log x \cdot 1 \right) = (1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x(1 + \log x).$$

EXAMPLE : $Differentiate\,(\sin x)^x$ $w.r.t.\ x.$

SOLUTION Let $y = (\sin x)^x$.

Taking logarithm on both sides of (i), we get

$$\log y = x \log (\sin x).$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}\{\log (\sin x)\} + \log (\sin x) \cdot \frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{\sin x} \cdot \cos x + \log (\sin x) \cdot 1$$

$$= x \cot x + \log (\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y \cdot [x \cot x + \log (\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x [x \cot x + \log (\sin x)].$$

EXAMPLE : *Differentiate* $(\sin x)^{\log x}$ *w.r.t. x.*

SOLUTION Let $y = (\sin x)^{\log x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (\log x)(\log \sin x). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (\log x)\cdot\frac{d}{dx}(\log \sin x) + (\log \sin x)\cdot\frac{d}{dx}(\log x)$$

$$= (\log x)\cdot\frac{1}{\sin x}\cdot\cos x + (\log \sin x)\cdot\frac{1}{x}$$

$$= (\log x)\cot x + \frac{(\log \sin x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left[(\log x)\cot x + \frac{(\log \sin x)}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\cdot\left[(\log x)\cot x + \frac{(\log \sin x)}{x}\right].$$

EXAMPLE : $If\ y=(x)^{\cos x}+(\cos x)^{\sin x}, find\ \frac{dy}{dx}$.

SOLUTION Let $y = u + v$, where $u = (x)^{\cos x}$ and $v = (\cos x)^{\sin x}$.

Now, $u = (x)^{\cos x}$

$$\Rightarrow \log u = (\cos x)(\log x)$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (\cos x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\cos x)$$

[on differentiating w.r.t. x]

$$= (\cos x) \cdot \frac{1}{x} + (\log x)(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}$$

$$\Rightarrow \frac{du}{dx} = (x)^{\cos x} \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}. \quad \text{... (i)}$$

And, $v = (\cos x)^{\sin x}$

$$\Rightarrow \log v = (\sin x) \log (\cos x)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = (\sin x) \cdot \frac{d}{dx} \{\log (\cos x)\} + \log (\cos x) \cdot \frac{d}{dx} (\sin x)$$

[on differentiating w.r.t. x]

$$\Rightarrow \frac{dv}{dx} = v \cdot \left\{ (\sin x) \cdot \frac{(-\sin x)}{\cos x} + \log (\cos x) \cdot \cos x \right\}$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}. \qquad \text{... (ii)}$$

$$\therefore \; y = (u + v)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (x)^{\cos x} \cdot \left\{ \frac{\cos x}{x} - (\log x) \sin x \right\}$$

$$+ (\cos x)^{\sin x} \cdot \{- \sin x \tan x + \cos x \cdot \log (\cos x)\}.$$

EXAMPLE : If $x^y = y^x$, find $\frac{dy}{dx}$.

EXAMPLE : If $x^y = y^x$, find $\frac{dy}{dx}$.

SOLUTION Given: $x^y = y^x$

$$\Rightarrow y \log x = x \log y. \qquad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$y \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(y) = x \cdot \frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow y \cdot \frac{1}{x} + (\log x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \cdot 1$$

$$\Rightarrow \left(\log x - \frac{x}{y}\right) \frac{dy}{dx} = \left(\log y - \frac{y}{x}\right)$$

$$\Rightarrow \frac{(y \log x - x)}{y} \cdot \frac{dy}{dx} = \frac{(x \log y - y)}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}.$$

EXAMPLE ʻ $\textit{If } x^y = e^{x-y}, \textit{ prove that } \dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}.$

SOLUTION We have

$$x^y = e^{x-y} \Rightarrow y \log x = (x-y)$$

$$\Rightarrow (1+\log x)y = x$$

$$\Rightarrow y = \frac{x}{(1+\log x)}. \qquad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot \frac{d}{dx}(x) - x\cdot \frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)\cdot 1 - x\cdot \frac{1}{x}}{(1+\log x)^2} = \frac{(1+\log x - 1)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}.$$

Differentiation of Implicit Function

EXAMPLE 1 If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$.

SOLUTION Given: $x^3 + y^3 = 3axy.$

Differentiating both sides of (i) w.r.t. x, we get

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a \cdot \left\{ x \cdot \frac{dy}{dx} + y \cdot 1 \right\}$$

$$\Rightarrow \; 3(y^2 - ax) \cdot \frac{dy}{dx} = 3(ay - x^2)$$

$$\Rightarrow \; \frac{dy}{dx} = \left(\frac{ay - x^2}{y^2 - ax} \right).$$

EXAMPLE 2 $If\ ax^2+2hxy+by^2+2gx+2fy+c=0,\quad find\ \frac{dy}{dx}.$

SOLUTION Given: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

Differentiating both sides of (i) w.r.t. x, we get

$$2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + 2by \cdot \frac{dy}{dx} + 2g + 2f \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow (2ax + 2hy + 2g) + (2hx + 2by + 2f) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{ax + hy + g}{hx + by + f}\right).$$

EXAMPLE 3 *If* $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, *prove that* $\dfrac{dy}{dx}=\sqrt{\dfrac{1-y^2}{1-x^2}}$.

SOLUTION Given: $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$.

Putting $x=\sin\theta$ and $y=\sin\phi$, it becomes

$$\cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$$

$$\Rightarrow \frac{\cos\theta+\cos\phi}{\sin\theta-\sin\phi}=a$$

$$\Rightarrow \frac{2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)}{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)} = a$$

$$\Rightarrow \cot\left(\frac{\theta-\phi}{2}\right) = a \Rightarrow \theta-\phi = 2\cot^{-1} a$$

$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a.$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0.$$

Hence, $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

EXAMPLE *If* $\sin y = x\sin(a+y)$, *prove that* $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$. [CBSE 2012]

SOLUTION $\sin y = x\sin(a+y)$

$$\Rightarrow \; x = \frac{\sin y}{\sin(a+y)}. \qquad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)} \quad \text{[using the quotient rule]}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}.$$

Hence, $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

EXAMPLE If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} = -e^{y-x}$.

SOLUTION Given: $e^x + e^y = e^{x+y}$.

On dividing throughout by e^{x+y}, we get

$$e^{-y} + e^{-x} = 1.$$

On differentiating both sides of (ii) w.r.t. x, we get

$$e^{-y} \cdot \left(\frac{-dy}{dx}\right) + e^{-x}(-1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}.$$

PRACTICE QUESTIONS

Find $\frac{dy}{dx}$, when:

1. $x^2 + y^2 = 4$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3. $\sqrt{x} + \sqrt{y} = \sqrt{a}$
4. $x^{2/3} + y^{2/3} = a^{2/3}$
5. $xy = c^2$
6. $x^2 + y^2 - 3xy = 1$
7. $xy^2 - x^2y - 5 = 0$
8. $(x^2 + y^2)^2 = xy$
9. $x^2 + y^2 = \log(xy)$

PRACTICE QUESTIONS

If $y\log x = (x-y)$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

If $\cos y = x\cos(y+a)$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(y+a)}{\sin a}$.

EXAMPLE   If $x^y + y^x = a^b$, find $\dfrac{dy}{dx}$.

ANSWERS

1. $\frac{-x}{y}$

2. $\frac{-b^2x}{a^2y}$

3. $-\sqrt{\frac{y}{x}}$

4. $\frac{-y^{1/3}}{x^{1/3}}$

5. $\frac{-c^2}{x^2}$

6. $\frac{(2x-3y)}{(3x-2y)}$

7. $\frac{(y^2-2xy)}{(x^2-2xy)}$

8. $\frac{(y-4xy^2-4x^3)}{(4y^3+4x^2y-x)}$

9. $\frac{y(1-2x^2)}{x(2y^2-1)}$

EXAMPLE . If $y = x^{x^{x \cdots \infty}}$, prove that $\dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y\log x)}$.

So, we may write the given function as $y = x^y$.

Now, $y = x^y \Rightarrow \log y = y \log x$.

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{(1 - y \log x)}{y} \cdot \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \left\{\frac{y}{x} \times \frac{y}{(1 - y \log x)}\right\} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}.$$

EXAMPLE : If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots \text{to } \infty}}}$, prove that $\frac{dy}{dx}=\frac{\cos x}{(2y-1)}$.

SOLUTION We may write the given series as

$$y = \sqrt{\sin x + y} \Rightarrow y^2 = (\sin x + y).$$

On differentiating both sides of (i) w.r.t. x, we get

$$2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y - 1) \cdot \frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(2y - 1)}.$$

EXAMPLE 3 If $y = e^{x+e^{x+e^{x+\dots \text{ to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{y}{(1-y)}$.

SOLUTION We may write the given series as

$$y = e^{x+y} \Rightarrow \log y = (x+y). \quad \dots \text{(i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{(1-y)}{y} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{(1-y)}.$$

EXAMPLE *Differentiate* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *w.r.t.* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

SOLUTION Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta,$$

$$v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta.$$

$$\therefore \quad u = v \Rightarrow \frac{du}{dv} = 1.$$

EXAMPLE *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *w.r.t.* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

SOLUTION Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta,$$

$$v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

$$\therefore \quad u = v \Rightarrow \frac{du}{dv} = 1.$$

EXAMPLE *Differentiate* $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ *w.r.t.* $\cos^{-1} x^2$

SOLUTION Let $u = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v = \cos^{-1}x^2$.

Then, $\cos^{-1}x^2 = v \Rightarrow x^2 = \cos v$.

Putting $x^2 = \cos v$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos v}-\sqrt{1-\cos v}}{\sqrt{1+\cos v}+\sqrt{1-\cos v}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(v/2)}-\sqrt{2\sin^2(v/2)}}{\sqrt{2\cos^2(v/2)}+\sqrt{2\sin^2(v/2)}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)}\right\} = \tan^{-1}\left\{\frac{1 - \tan(v/2)}{1 + \tan(v/2)}\right\}$$

[dividing num. and denom. by $\cos(v/2)$]

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{v}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{v}{2}\right).$$

$$\therefore \; u = \left(\frac{\pi}{4} - \frac{v}{2}\right) \Rightarrow \frac{du}{dv} = \frac{-1}{2}.$$

. Derivatives of Parametric Functions

Sometimes x and y are given as functions of a variable t. Then, t is called a *parameter*.

Let $x = f(t)$ and $y = g(t)$. Then,

$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = g'(t).$$

$$\therefore \quad \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$

SUMMARY

Let $x = f(t)$ and $y = g(t)$. Then,

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$

EXAMPLE 1 Find $\frac{dy}{dx}$, when $x = a(t + \sin t)$ and $y = a(1 - \cos t)$.

SOLUTION We have:

$$x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t);$$

$$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t.$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{a \sin t}{a(1 + \cos t)} = \frac{2a \sin(t/2) \cos(t/2)}{2a \cos^2(t/2)} = \tan\frac{t}{2}$$

EXAMPLE 2 $If\ x = a[\cos\theta + \log\ \tan(\theta/2)]\ and\ y = a\sin\theta, find\ \frac{dy}{dx}\ at\ \theta = (\pi/4).$

SOLUTION We have

$$x = a\{\cos\theta + \log \tan(\theta/2)\}$$

$$\Rightarrow \frac{dx}{d\theta} = a\left\{-\sin\theta + \frac{\sec^2(\theta/2)}{2\tan(\theta/2)}\right\} = a\left\{-\sin\theta + \frac{1}{2\sin(\theta/2)\cos(\theta/2)}\right\}$$

$$= a\left\{-\sin\theta + \frac{1}{\sin\theta}\right\} = \frac{a(1-\sin^2\theta)}{\sin\theta} = \frac{a\cos^2\theta}{\sin\theta}.$$

And, $y = a\sin\theta \Rightarrow \dfrac{dy}{d\theta} = a\cos\theta.$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{d\theta} \times \frac{d\theta}{dx}\right) = \left(a\cos\theta \cdot \frac{\sin\theta}{a\cos^2\theta}\right) = \tan\theta.$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{\theta=\pi/4} = \tan\frac{\pi}{4} = 1.$$

- ***THANK YOU***