

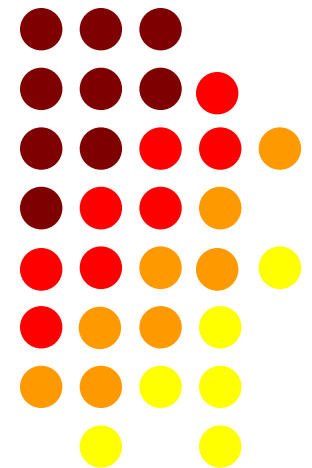
■ UNIT-4



Lecture- 29



Introduction of Limit



LIMIT We say that $\lim_{x \rightarrow a} f(x) = l$ if whenever $x \rightarrow a$, $f(x) \rightarrow l$.

Working rules for finding $\lim_{x \rightarrow a} f(x)$

RULE 1 Put $x = a$ in the given function. If $f(a)$ is a definite value then

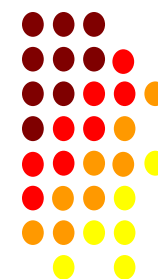
$$\lim_{x \rightarrow a} f(x) = f(a).$$

EXAMPLES 1. $\lim_{x \rightarrow 0} \sin x = \sin 0 = 0$.

2. $\lim_{x \rightarrow 1} (x^2 + 5x - 2) = (1^2 + 5 \times 1 - 2) = 4$.



RULE II If $f(x)$ is a rational function then factorize the numerator and the denominator. Cancel out the common factors and then put $x = a$.



EXAMPLE Evaluate (i) $\lim_{x \rightarrow 3} \left(\frac{x^2 - 9}{x - 3} \right)$ (ii) $\lim_{x \rightarrow 1} \frac{(x^2 - 4x + 3)}{(x - 1)}$.

SOLUTION (i) $\lim_{x \rightarrow 3} \left(\frac{x^2 - 9}{x - 3} \right) = \lim_{x \rightarrow 3} \frac{(x + 3)(x - 3)}{(x - 3)} = \lim_{x \rightarrow 3} (x + 3) = 6.$

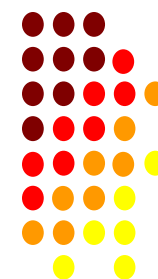
(ii) $\lim_{x \rightarrow 1} \frac{(x^2 - 4x + 3)}{(x - 1)} = \lim_{x \rightarrow 1} \frac{(x - 1)(x - 3)}{(x - 1)} = \lim_{x \rightarrow 1} (x - 3) = -2.$



EXAMPLE

Evaluate

$$\lim_{x \rightarrow 0} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \right\}$$



SOLUTION We have

$$\lim_{x \rightarrow 0} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \right\} = \lim_{x \rightarrow 0} \left\{ \frac{(\sqrt{1+x} - \sqrt{1-x}) \cdot (\sqrt{1+x} + \sqrt{1-x})}{x \cdot (\sqrt{1+x} + \sqrt{1-x})} \right\}$$



$$= \lim_{x \rightarrow 0} \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \rightarrow 0} \frac{2}{(\sqrt{1+x} + \sqrt{1-x})} = 1$$

[putting $x = 0$].



Fundamental Theorems on Limits (without proof)

$$(i) \lim_{x \rightarrow a} \{f(x) + g(x)\} = \left\{ \lim_{x \rightarrow a} f(x) \right\} + \left\{ \lim_{x \rightarrow a} g(x) \right\}$$

$$(ii) \lim_{x \rightarrow a} \{f(x) - g(x)\} = \left\{ \lim_{x \rightarrow a} f(x) \right\} - \left\{ \lim_{x \rightarrow a} g(x) \right\}$$

$$(iii) \lim_{x \rightarrow a} \{c \cdot f(x)\} = c \cdot \left\{ \lim_{x \rightarrow a} f(x) \right\}, \text{ where } c \text{ is a constant}$$

$$(iv) \lim_{x \rightarrow a} \{f(x) \cdot g(x)\} = \left\{ \lim_{x \rightarrow a} f(x) \right\} \cdot \left\{ \lim_{x \rightarrow a} g(x) \right\}$$



$$(v) \lim_{x \rightarrow a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\left\{ \lim_{x \rightarrow a} f(x) \right\}}{\left\{ \lim_{x \rightarrow a} g(x) \right\}}, \text{ provided } \lim_{x \rightarrow a} g(x) \neq 0$$



SUMMARY

$$1. \lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1}, \text{ where } a > 0$$

$$2. \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = 1.$$

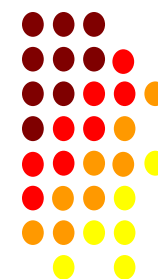
$$3. \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \log_e a.$$

$$4. \lim_{x \rightarrow 0} (1 + x)^{1/x} = e.$$

$$5. \lim_{x \rightarrow 0} \frac{\log(1 + x)}{x} = 1.$$



EXAMPLE 1 Evaluate $\lim_{x \rightarrow a} \left\{ \frac{x^{12} - a^{12}}{x - a} \right\}$.



EXAMPLE . Evaluate $\lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$.

SOLUTION $\lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

$$= \lim_{(x+2) \rightarrow (a+2)} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{(x+2) - (a+2)}$$

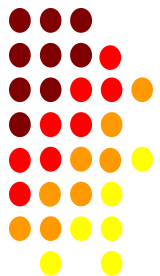
$$= \frac{3}{2} \cdot (a+2)^{\left(\frac{3}{2}-1\right)} = \frac{3}{2} (a+2)^{1/2}$$

$$\left[\because \lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x-a} \right) = na^{n-1} \right]$$



Evaluate:

$$(i) \lim_{x \rightarrow 0} \left(\frac{e^{-x} - 1}{x} \right) \quad (ii) \lim_{x \rightarrow 0} \left(\frac{e^x - e^{-x}}{x} \right) \quad (iii) \lim_{x \rightarrow 0} \left(\frac{e^x + e^{-x} - 2}{x^2} \right)$$



EXAMPLE : Evaluate $\lim_{x \rightarrow 0} \left(\frac{3^x - 2^x}{x} \right)$.



$$\begin{aligned}\text{SOLUTION } \lim_{x \rightarrow 0} \left(\frac{3^x - 2^x}{x} \right) &= \lim_{x \rightarrow 0} \left\{ \frac{(3^x - 1) - (2^x - 1)}{x} \right\} \\ &= \lim_{x \rightarrow 0} \left(\frac{3^x - 1}{x} \right) - \lim_{x \rightarrow 0} \left(\frac{2^x - 1}{x} \right) \\ &= (\log 3 - \log 2) = \log \frac{3}{2} \quad \left[\because \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \log a \right]\end{aligned}$$



TRIGONOMETRIC LIMITS



$$(i) \lim_{\theta \rightarrow 0} \sin \theta = 0$$

$$(ii) \lim_{\theta \rightarrow 0} \cos \theta = 1$$



THEOREM 2 (i) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$



EXAMPLE 1 Evaluate:

$$(i) \lim_{x \rightarrow 0} \frac{\sin 2x}{x}$$

$$(ii) \lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$$

$$(iii) \lim_{x \rightarrow 0} \left(\frac{\sin ax}{\sin bx} \right)$$



EXAMPLE 2 Evaluate $\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x}$.



EXAMPLE Evaluate:

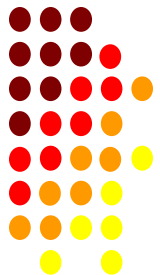
$$(i) \lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) \qquad (ii) \lim_{x \rightarrow 0} \frac{(\operatorname{cosec} x - \cot x)}{x}$$

SOLUTION (i) $\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x) \cos x}{\cos^2 x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x) \cos x}{(1 - \sin^2 x)}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)} = 0 \qquad \left[\text{putting } x = \frac{\pi}{2} \right]$$



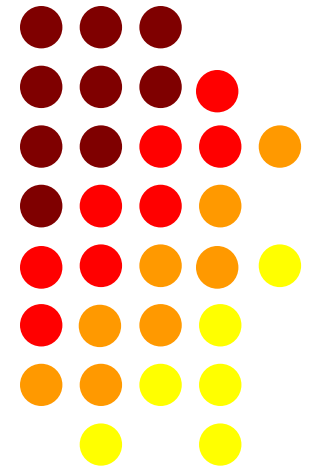


THANK YOU



Lecture-30

Continuity



Continuity

CONTINUITY AT A POINT A real function $f(x)$ is said to be continuous at a point a of its domain if $\lim_{x \rightarrow a} f(x)$ exists and equals $f(a)$.

Thus, $f(x)$ is continuous at $x = a$ if

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a).$$

If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.



If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.

REMARK $f(x)$ is discontinuous at $x = a$ in each of the following cases:

(i) $f(a)$ is not defined

(ii) $\lim_{x \rightarrow a} f(x)$ does not exist

(iii) $\lim_{x \rightarrow a} f(x) \neq f(a)$



EXAMPLE 1 Show that $f(x) = x^3$ is continuous at $x = 2$.



EXAMPLE 1 Show that $f(x) = x^3$ is continuous at $x = 2$.

SOLUTION We have $f(2) = 2^3 = 8$;

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} (2+h)^3 = \lim_{h \rightarrow 0} (8 + h^3 + 12h + 6h^2) = 8;$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} (2-h)^3 = \lim_{h \rightarrow 0} (8 - h^3 - 12h + 6h^2) = 8.$$

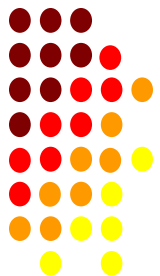
$$\therefore \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = f(2).$$

Hence, $f(x)$ is continuous at $x = 2$.



EXAMPLE . Discuss the continuity of the function $f(x)$ at $x = 0$, if

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0. \end{cases}$$



SOLUTION Clearly, $f(0) = (2 \times 0 + 1) = 1$.

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) \\ &= \lim_{h \rightarrow 0} [2(0 + h) + 1] = \lim_{h \rightarrow 0} (2h + 1) = 1.\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) \\ &= \lim_{h \rightarrow 0} [2(0 - h) - 1] = \lim_{h \rightarrow 0} (-2h - 1) = -1.\end{aligned}$$

Thus, $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$ and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.



EXAMPLE . Show that the function $f(x) = \begin{cases} 3x - 2, & \text{when } x \leq 0 \\ x + 1, & \text{when } x > 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION We have, $f(0) = (3 \times 0 - 2) = -2$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) \\ &= \lim_{h \rightarrow 0} (h + 1) = 1. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) \\ &= \lim_{h \rightarrow 0} [3(-h) - 2] = \lim_{h \rightarrow 0} (-3h - 2) = -2. \end{aligned}$$

$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$ and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.



EXAMPLE : Show that the function $f(x) = \begin{cases} 3x - 2, & \text{when } x \leq 0 \\ x + 1, & \text{when } x > 0 \end{cases}$
is discontinuous at $x = 0$.



SOLUTION We have, $f(0) = (3 \times 0 - 2) = -2$.

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} (h+1) = 1.\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\ &= \lim_{h \rightarrow 0} [3(-h) - 2] = \lim_{h \rightarrow 0} (-3h - 2) = -2.\end{aligned}$$

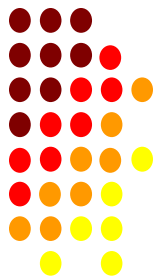
$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$ and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.



EXAMPLE Show that the function $f(x) = \begin{cases} \frac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.



EXAMPLE . Show that the function $f(x) = \begin{cases} \frac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION It is being given that $f(0) = 1$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{h}{|h|} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{-h}{|h|} = \lim_{h \rightarrow 0} \frac{-h}{h} = -1.$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x).$$

So, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.



EXAMPLE *Examine the continuity of the function*

$$f(x) = \begin{cases} \frac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \text{ at } x = 0.$$



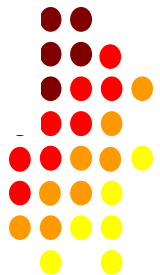
SOLUTION We have $f(0) = 1$.

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} \frac{|\sin(0+h)|}{(0+h)} = \lim_{h \rightarrow 0} \frac{|\sin h|}{h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0-h) \\ &= \lim_{h \rightarrow 0} \frac{|\sin(-h)|}{-h} = \lim_{h \rightarrow 0} \frac{|-\sin h|}{-h} = \lim_{h \rightarrow 0} \frac{\sin h}{-h} = -1.\end{aligned}$$

$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$. So, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.



EXAMPLE

Find the value of k for which

$$f(x) = \begin{cases} kx + 5, & \text{when } x \leq 2 \\ x - 1, & \text{when } x > 2 \end{cases}$$

is continuous at $x = 2$.



SOLUTION We have, $f(2) = (k \times 2 + 5) = (2k + 5)$.

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h)$$



$$\begin{aligned} &= \lim_{h \rightarrow 0} \{(2 + h) - 1\} = \lim_{h \rightarrow 0} (1 + h) = 1. \\ \lim_{x \rightarrow 2^-} f(x) &= \lim_{h \rightarrow 0} f(2 - h) \\ &= \lim_{h \rightarrow 0} \{k(2 - h) + 5\} = \lim_{h \rightarrow 0} \{(2k + 5) - kh\} = (2k + 5). \end{aligned}$$

Now, $\lim_{x \rightarrow 2} f(x)$ exists only when $2k + 5 = 1$, i.e., when $k = -2$.

When $k = -2$, we have $\lim_{x \rightarrow 2} f(x) = f(2) = 1$.

Hence, $f(x)$ is continuous at $x = 2$ when $k = -2$.



EXAMPLE † If the function $f(x) = \begin{cases} 3ax + b, & \text{for } x > 1 \\ 11, & \text{for } x = 1 \\ 5ax - 2b, & \text{for } x < 1 \end{cases}$

is continuous at $x = 1$, find the values of a and b .



SOLUTION We have, $f(1) = 11$.

$$\begin{aligned}\lim_{x \rightarrow 1^+} f(x) &= \lim_{h \rightarrow 0} f(1+h) \\ &= \lim_{h \rightarrow 0} \{3a(1+h) + b\} = \lim_{h \rightarrow 0} \{(3a+b) + 3ah\} \\ &= (3a+b).\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{h \rightarrow 0} f(1-h) \\ &= \lim_{h \rightarrow 0} \{5a(1-h) - 2b\} = \lim_{h \rightarrow 0} \{(5a-2b) - 5ah\} \\ &= (5a-2b).\end{aligned}$$

Since $f(x)$ is continuous at $x = 1$, we have

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1).$$

$$\therefore 3a + b = 5a - 2b = 11.$$

On solving $(3a + b = 11)$ and $(5a - 2b = 11)$, we get $a = 3$, $b = 2$.

Hence, $a = 3$, $b = 2$.



EXAMPLE . Show that the function $f(x) = \begin{cases} \left(\frac{e^{1/x} - 1}{e^{1/x} + 1} \right), & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.



SOLUTION Clearly, $f(0) = 0$.

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0} f(x) &= \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1} \right) \\ &= \lim_{h \rightarrow 0} \frac{e^{1/h} \left(1 - \frac{1}{e^{1/h}} \right)}{e^{1/h} \left(1 + \frac{1}{e^{1/h}} \right)} = \lim_{h \rightarrow 0} \frac{\left(1 - \frac{1}{e^{1/h}} \right)}{\left(1 + \frac{1}{e^{1/h}} \right)} = 1. \end{aligned}$$

$$\begin{aligned} \text{And, } \lim_{x \rightarrow 0} f(x) &= \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} \left(\frac{e^{-1/h} - 1}{e^{-1/h} + 1} \right) \\ &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{e^{1/h}} - 1 \right)}{\left(\frac{1}{e^{1/h}} + 1 \right)} = -1. \end{aligned}$$

Thus, $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$, and therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.



PRACTICE QUESTIONS

Prove That

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3; \\ 5, & \text{when } x = 3 \end{cases} \text{ is continuous at } x = 3.$$

$$f(x) = \begin{cases} \frac{x^2 - 25}{x - 5}, & \text{when } x \neq 5; \\ 10, & \text{when } x = 5 \end{cases} \text{ is continuous at } x = 5.$$

$$f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x = 0.$$

$$f(x) = \begin{cases} \frac{1 - \cos x}{x^2}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x = 0.$$

$$f(x) = \begin{cases} 2 - x, & \text{when } x < 2; \\ 2 + x, & \text{when } x \geq 2 \end{cases} \text{ is discontinuous at } x = 2.$$

$$f(x) = \begin{cases} 3 - x, & \text{when } x \leq 0; \\ x^2, & \text{when } x > 0 \end{cases} \text{ is discontinuous at } x = 0.$$



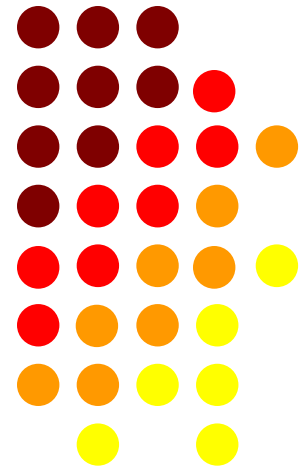


THANK YOU



Lecture- 31

Differentiability



Differentiability

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) Right-hand derivative $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Rf'(a)$.

(ii) Left-hand derivative $\lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Lf'(a)$.

DIFFERENTIABILITY A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.



EXAMPLE 1 Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.



EXAMPLE 1 Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.

SOLUTION
$$Rf'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{1 + h^2 + 2h - 1}{h} \right) = \lim_{h \rightarrow 0} (h + 2) = 2.$$

And,
$$Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{(1-h)^2 - (1)^2}{-h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{1 + h^2 - 2h - 1}{-h} \right) = \lim_{h \rightarrow 0} (-h + 2) = 2.$$

$\therefore Rf'(1) = Lf'(1) = 2.$

This shows that $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$.



EXAMPLE : Show that the function $f(x) = \begin{cases} 1 + x, & \text{if } x \leq 2; \\ 5 - x, & \text{if } x > 2 \end{cases}$ is not differentiable at $x = 2$.



SOLUTION
$$Rf'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \left[\frac{5 - (2+h) - 3}{h} \right]$$
$$= \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} (-1) = -1.$$

And,
$$Lf'(2) = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h}$$
$$= \lim_{h \rightarrow 0} \left[\frac{1 + (2-h) - 3}{-h} \right] = \lim_{h \rightarrow 0} \frac{-h}{-h} = \lim_{h \rightarrow 0} 1 = 1.$$

Thus, $Rf'(2) \neq Lf'(2)$.

Hence, $f(x)$ is not differentiable at $x = 2$.



THEOREM *Every differentiable function is continuous. But, every continuous function need not be differentiable.*



EXAMPLE 8 Show that $f(x) = |x - 2|$ is continuous but not differentiable at $x = 2$.



SOLUTION We have $f(2) = |2 - 2| = 0$.

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} |2+h-2| = \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0.$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} |2-h-2| = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h = 0.$$

$$\therefore \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = f(2) = 0.$$

So, $f(x)$ is continuous at $x = 2$.



But,

$$Rf'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{|2+h-2| - 0}{h}$$
$$= \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

And,

$$Lf'(2) = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \rightarrow 0} \frac{|2-h-2| - 0}{-h}$$
$$= \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1.$$

Thus, $Rf'(2) \neq Lf'(2)$.

This shows that $f(x)$ is not differentiable at $x = 2$.



PRACTICE QUESTIONS

Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

$$\text{Let } f(x) = \begin{cases} (2-x), & \text{when } x \geq 1 \\ x, & \text{when } 0 \leq x < 1. \end{cases}$$

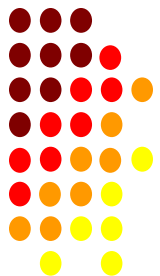
Show that $f(x)$ is continuous but not differentiable at $x = 1$.

Show that $f(x) = [x]$ is neither continuous nor derivable at $x = 2$.

$$\text{Show that the function } f(x) = \begin{cases} (1-x), & \text{when } x < 1; \\ (x^2 - 1), & \text{when } x \geq 1 \end{cases}$$

is continuous but not differentiable at $x = 1$.

$$\text{Let } f(x) = \begin{cases} (2+x), & \text{if } x \geq 0 \\ (2-x), & \text{if } x < 0. \end{cases} \quad \text{Show that } f(x) \text{ is not derivable at } x = 0.$$





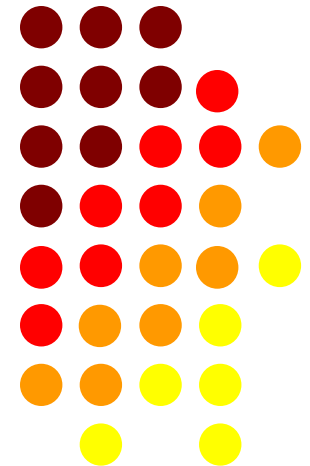
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Lecture- 32 & 33



Derivative of sum,difference, and
its based questions



SUMMARY

We may summarise the above results as given below:

$$(i) \frac{d}{dx} (x^n) = nx^{n-1}$$

$$(ii) \frac{d}{dx} (e^x) = e^x$$

$$(iii) \frac{d}{dx} (\sin x) = \cos x$$

$$(iv) \frac{d}{dx} (\cos x) = -\sin x$$

$$(v) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$(vi) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$(vii) \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$(viii) \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x.$$

The derivative of a constant function is zero, i.e., $\frac{d}{dx} (c) = 0$.



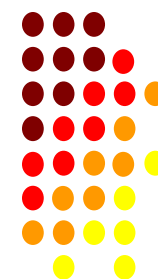
EXAMPLE 1 Find the derivative of

(i) $8x^3$

(ii) $6\sqrt{x}$

(iii) $5e^x$

(iv) 9×2^x



EXAMPLE 2 Find the derivative of $(x^3 + e^x + 3^x + \cot x)$ with respect to x .



EXAMPLE 3 Find the derivative of $\left(9x^2 + \frac{3}{x} + 5 \sin x\right)$ with respect to x .



SOLUTION We have $\frac{d}{dx} \left(9x^2 + \frac{3}{x} + 5 \sin x \right)$

$$= 9 \cdot \frac{d}{dx} (x^2) + 3 \cdot \frac{d}{dx} (x^{-1}) + 5 \cdot \frac{d}{dx} (\sin x)$$
$$= 9 \times 2x + 3 \cdot (-1) x^{-2} + 5 \cos x = 18x - \frac{3}{x^2} + 5 \cos x.$$



EXAMPLE 1 Differentiate the following functions with respect to x :

$$\left(x^2 + \frac{4}{x^2} - \frac{2}{3} \tan x + 6e \right)$$



SOLUTION $\frac{d}{dx} \left(x^2 + \frac{4}{x^2} - \frac{2}{3} \tan x + 6e \right)$

$$= \frac{d}{dx} (x^2) + 4 \cdot \frac{d}{dx} (x^{-2}) - \frac{2}{3} \cdot \frac{d}{dx} (\tan x) + 6 \cdot \frac{d}{dx} (e)$$

$$= 2x + 4 \cdot (-2) x^{-3} - \frac{2}{3} \sec^2 x + 6 \times 0 \quad \left[\because \frac{d}{dx} (e) = 0 \right]$$

$$= 2x - \frac{8}{x^3} - \frac{2}{3} \sec^2 x.$$

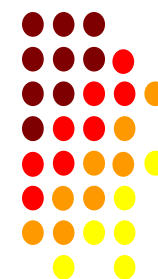


EXAMPLE 2 Find the derivative of $\left\{ \frac{3}{\sqrt[3]{x}} - \frac{5}{\cos x} + \frac{6}{\sin x} - \frac{2 \tan x}{\sec x} + 7 \right\}$.



EXAMPLE 3 Differentiate the following functions:

(i) $(x^2 - 5x + 6)(x - 3)$ (ii) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (iii) $\frac{3x^2 + 2x + 5}{\sqrt{x}}$



EXAMPLE 4 If $y = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$, find $\frac{dy}{dx}$.

SOLUTION $y = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} = \sqrt{\frac{2 \sin^2 x}{2 \cos^2 x}} = \tan x.$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\tan x) = \sec^2 x.$$



EXAMPLE 5 If $y = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty \right)$, show that $\frac{dy}{dx} = y$.

SOLUTION We have, $y = e^x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^x) = e^x = y.$$



EXAMPLE 6 If $u = 3t^4 - 5t^3 + 2t^2 - 18t + 4$, find $\frac{du}{dt}$ at $t = 1$.



SOLUTION

$$\begin{aligned}\frac{du}{dt} &= \frac{d}{dt} (3t^4 - 5t^3 + 2t^2 - 18t + 4) \\ &= 3 \cdot \frac{d}{dt} (t^4) - 5 \cdot \frac{d}{dt} (t^3) + 2 \cdot \frac{d}{dt} (t^2) - 18 \cdot \frac{d}{dt} (t) + \frac{d}{dt} (4) \\ &= 3 \times 4t^3 - 5 \times 3t^2 + 2 \times 2t - 18 \times 1 + 0 \\ &= 12t^3 - 15t^2 + 4t - 18.\end{aligned}$$
$$\begin{aligned}\therefore \left(\frac{du}{dt} \right)_{t=1} &= (12 \times 1^3 - 15 \times 1^2 + 4 \times 1 - 18) \\ &= (12 - 15 + 4 - 18) = -17.\end{aligned}$$



DERIVATIVE OF THE PRODUCT OF FUNCTIONS

THEOREM (Product rule) *If $f(x)$ and $g(x)$ are two differentiable functions then $f(x) \cdot g(x)$ is also differentiable, and*

$$\frac{d}{dx} \{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx} \{g(x)\} + g(x) \cdot \frac{d}{dx} \{f(x)\}.$$



EXAMPLE 1

Differentiate: (i) xe^x

(ii) $x^2 e^x \sin x$



EXAMPLE 2 *Differentiate* $x^2 \tan x$.



PRACTICE QUESTIONS

Differentiate:

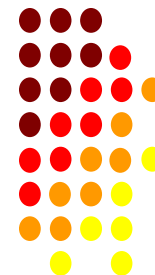
$$(x^2 + 3x + 1) \sin x$$

$$x^N \cot x$$





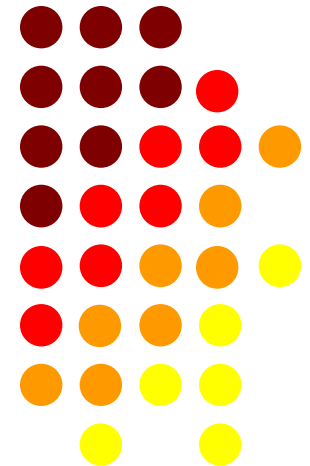
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Lecture- 34,35



Derivative of Quotient of functions and Composite Functions



Derivative of the Quotient of two Functions

THEOREM (Quotient rule) If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$ then $\frac{f(x)}{g(x)}$ is also differentiable, and

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) \cdot \frac{d}{dx} \{f(x)\} - f(x) \cdot \frac{d}{dx} \{g(x)\}}{[g(x)]^2} .$$



EXAMPLE 1 Differentiate:

(i) $\frac{e^x}{x}$

(ii) $\left(\frac{2x + 3}{x^2 - 5}\right)$

(iii) $\frac{e^x}{(1 + \sin x)}$



EXAMPLE 2 Differentiate $\left(\frac{x^2 + 5x - 6}{4x^2 - x + 3} \right)$.



EXAMPLE. If $y = \left\{ \frac{1 - \tan x}{1 + \tan x} \right\}$, show that $\frac{dy}{dx} = \frac{-2}{(1 + \sin 2x)}$.



SOLUTION By the quotient rule, we have

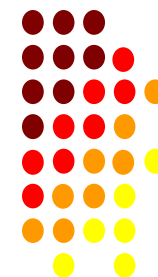
$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(1 + \tan x) \cdot \frac{d}{dx}(1 - \tan x) - (1 - \tan x) \cdot \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\
 &= \frac{(1 + \tan x)(-\sec^2 x) - (1 - \tan x)(\sec^2 x)}{(1 + \tan x)^2} \\
 &= \frac{-2\sec^2 x}{(1 + \tan x)^2} = \frac{-2}{(\cos^2 x)(1 + \tan^2 x + 2\tan x)} \\
 &= \frac{-2}{(\cos^2 x) \left\{ 1 + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos x} \right\}} = \frac{-2}{(1 + \sin 2x)}.
 \end{aligned}$$



EXAMPLE 5 Differentiate:

$$(i) \left(\frac{\sin x + \cos x}{\sin x - \cos x} \right)$$

$$(ii) \left(\frac{\sec x - 1}{\sec x + 1} \right)$$



SOLUTION (i) $\frac{d}{dx} \left(\frac{\sin x + \cos x}{\sin x - \cos x} \right)$

$$= \frac{(\sin x - \cos x) \cdot \frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x) \cdot \frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-[(\sin x - \cos x)^2 + (\sin x + \cos x)^2]}{(\sin x - \cos x)^2}$$

$$= \frac{-2(\sin^2 x + \cos^2 x)}{(\sin^2 x + \cos^2 x - 2\sin x \cos x)} = \frac{-2}{(1 - \sin 2x)}$$



$$\begin{aligned} \text{(ii)} \quad & \frac{d}{dx} \left(\frac{\sec x - 1}{\sec x + 1} \right) \\ &= \frac{(\sec x + 1) \cdot \frac{d}{dx} (\sec x - 1) - (\sec x - 1) \cdot \frac{d}{dx} (\sec x + 1)}{(\sec x + 1)^2} \\ &= \frac{(\sec x + 1) \sec x \tan x - (\sec x - 1) \sec x \tan x}{(\sec x + 1)^2} = \frac{2 \sec x \tan x}{(\sec x + 1)^2} \end{aligned}$$



PRACTICE QUESTIONS

Differentiate:

1. $\frac{2^x}{x}$

2. $\frac{\log x}{x}$

3. $\frac{e^x}{(1+x)}$

4. $\frac{e^x}{(1+x^2)}$

5. $\left(\frac{2x^2-4}{3x^2+7}\right)$

6. $\left(\frac{x^2+3x-1}{x+2}\right)$

7. $\frac{(x^2-1)}{(x^2+7x+1)}$

8. $\left(\frac{5x^2+6x+7}{2x^2+3x+4}\right)$

9. $\frac{x}{(a^2+x^2)}$

10. $\frac{x^4}{\sin x}$

11. $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

12. $\frac{\cos x}{\log x}$



ANSWERS

1. $\frac{2^x(x \log 2 - 1)}{x^2}$

2. $\frac{(1 - \log x)}{x^2}$

3. $\frac{xe^x}{(1+x)^2}$

4. $\frac{e^x(1-x)^2}{(1+x^2)^2}$

5. $\frac{52x}{(3x^2+7)^2}$

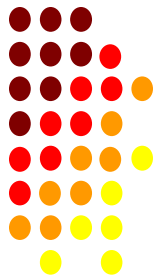
6. $\frac{(x^2+4x+7)}{(x+2)^2}$

7. $\frac{(7x^2+4x+7)}{(x^2+7x+1)^2}$

8. $\frac{3(x^2+4x+1)}{(2x^2+3x+4)^2}$

9. $\frac{(a^2-x^2)}{(a^2+x^2)^2}$

10. $\frac{x^3(4 \sin x - x \cos x)}{\sin^2 x}$



ANSWERS

$$11. \frac{\sqrt{a}}{\sqrt{x} \cdot (\sqrt{a} - \sqrt{x})^2}$$

$$12. \frac{-(x \sin x \log x + \cos x)}{x(\log x)^2}$$



Derivative of Quotient of functions and Composite Functions



Derivative of a Function of a Function

CHAIN RULE If $y = f(t)$ and $t = g(x)$ then $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right)$.

This rule may be extended further.

If $y = f(t)$, $t = g(u)$ and $u = h(x)$ then $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx} \right)$.



EXAMPLE 1 Differentiate (i) $\sin x^3$

(ii) $\sin^3 x$

(iii) $e^{\sin x}$



EXAMPLE 2 If $y = \frac{1}{\sqrt{a^2 - x^2}}$, find $\frac{dy}{dx}$.



SOLUTION Put $(a^2 - x^2) = t$, so that $y = \frac{1}{\sqrt{t}} = t^{-1/2}$ and $t = (a^2 - x^2)$.

$$\therefore \frac{dy}{dt} = -\frac{1}{2}t^{-3/2} \quad \text{and} \quad \frac{dt}{dx} = -2x.$$

$$\begin{aligned} \text{So, } \frac{dy}{dx} &= \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) \\ &= \left(-\frac{1}{2}t^{-3/2} \right) (-2x) = xt^{-3/2} = x(a^2 - x^2)^{-3/2}. \end{aligned}$$

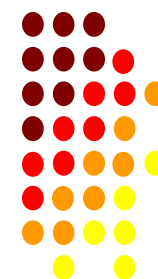


EXAMPLE 3

Differentiate:

(i) $(ax + b)^m$

(ii) $(3x + 5)^6$



EXAMPLE Differentiate $e^{\sqrt{\cot x}}$.

SOLUTION Let $y = e^{\sqrt{\cot x}}$. Put $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, so that

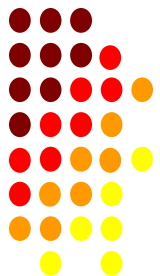
$$y = e^u, \quad u = \sqrt{t} \quad \text{and} \quad t = \cot x.$$

$$\therefore \frac{dy}{du} = e^u, \quad \frac{du}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \quad \text{and} \quad \frac{dt}{dx} = -\operatorname{cosec}^2 x.$$

$$\text{So, } \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right) = -\frac{1}{2} \cdot \frac{\operatorname{cosec}^2 x}{\sqrt{t}} e^u$$

$$= \frac{-\operatorname{cosec}^2 x}{2\sqrt{t}} \cdot e^{\sqrt{t}} \quad [\because u = \sqrt{t}]$$

$$= \frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}} \cdot e^{\sqrt{\cot x}} \quad [\because t = \cot x].$$



EXAMPLE If $y = \cos^2 x^2$, find $\frac{dy}{dx}$.

SOLUTION $y = (\cos x^2)^2$. Put $x^2 = t$ and $\cos x^2 = \cos t = u$, so that

$$y = u^2, \quad u = \cos t \quad \text{and} \quad t = x^2.$$

$$\therefore \frac{dy}{du} = 2u, \quad \frac{du}{dt} = -\sin t \quad \text{and} \quad \frac{dt}{dx} = 2x.$$

$$\begin{aligned} \text{So, } \frac{dy}{dx} &= \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right) \\ &= -4ux \sin t = -4x \sin t \cos t \quad [\because u = \cos t] \\ &= -4x \sin x^2 \cos x^2 = -2x \sin(2x^2) \quad [\because t = x^2] \end{aligned}$$



EXAMPLE

Differentiate

$$\sqrt{\frac{1 - \tan x}{1 + \tan x}}$$



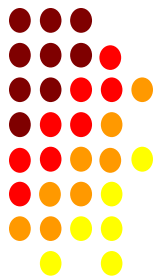
EXAMPLE If $y = \sin(\sqrt{\sin x + \cos x})$, find $\frac{dy}{dx}$.

SOLUTION Putting $(\sin x + \cos x) = t$ and $\sqrt{(\sin x + \cos x)} = \sqrt{t} = u$, we get
 $y = \sin u$, $u = \sqrt{t}$ and $t = (\sin x + \cos x)$.

$$\therefore \frac{dy}{du} = \cos u, \quad \frac{du}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}$$

$$\text{and } \frac{dt}{dx} = (\cos x - \sin x).$$

$$\begin{aligned} \text{So, } \frac{dy}{dx} &= \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right) = \frac{\cos u}{2\sqrt{t}} \cdot (\cos x - \sin x) \\ &= \frac{\cos \sqrt{t}}{2\sqrt{t}} \cdot (\cos x - \sin x) \quad [\because u = \sqrt{t}] \\ &= \frac{\cos(\sqrt{\sin x + \cos x})(\cos x - \sin x)}{2\sqrt{\sin x + \cos x}} \quad [\because t = (\sin x + \cos x)]. \end{aligned}$$



EXAMPLE! Differentiate $e^{ax} \cos(bx + c)$.

SOLUTION Using the product rule, we have

$$\begin{aligned} & \frac{d}{dx} (e^{ax} \cos(bx + c)) \\ &= e^{ax} \cdot \frac{d}{dx} \{\cos(bx + c)\} + \cos(bx + c) \cdot \frac{d}{dx} (e^{ax}) \\ &= e^{ax} \{-\sin(bx + c)\} \cdot \frac{d}{dx} (bx + c) + \cos(bx + c) \cdot e^{ax} \cdot \frac{d}{dx} (ax) \\ & \hspace{15em} \text{[using the chain rule]} \\ &= -be^{ax} \sin(bx + c) + ae^{ax} \cos(bx + c) \\ &= e^{ax} [a \cos(bx + c) - b \sin(bx + c)]. \end{aligned}$$



PRACTICE QUESTIONS

1. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, show that $\frac{dy}{dx} + y^2 + 1 = 0$.

2. If $y = \frac{\cos x + \sin x}{\cos x - \sin x}$, show that $\frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$.

3. If $y = \sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx} + y = 0$.

4. If $y = \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}}$, show that $\frac{dy}{dx} = \sec x (\tan x + \sec x)$.



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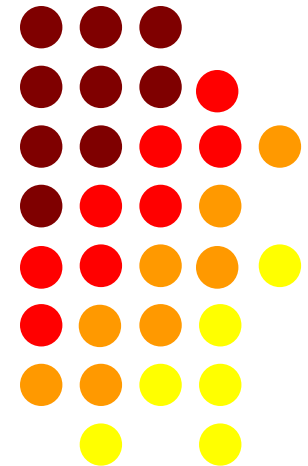
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Lecture- 36



Derivatives of Inverse Trigonometric Function



SUMMARY

$$(i) \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

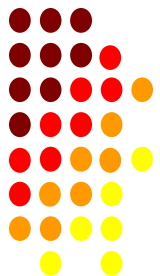
$$(ii) \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$(iii) \frac{d}{dx} (\tan^{-1} x) = \frac{1}{(1+x^2)}$$

$$(iv) \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{(1+x^2)}$$

$$(v) \frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}}$$

$$(vi) \frac{d}{dx} (\operatorname{cosec}^{-1} x) = \frac{-1}{|x| \sqrt{x^2-1}}$$



EXAMPLE 1 Differentiate the following w.r.t. x :

(i) $\sin^{-1} 2x$ (ii) $\tan^{-1} \sqrt{x}$ (iii) $\cos^{-1}(\cot x)$



SOLUTION

(i) Let $y = \sin^{-1}2x$.

Putting $2x = t$, we get $y = \sin^{-1}t$ and $t = 2x$.

$$\text{Now, } y = \sin^{-1}t \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}.$$

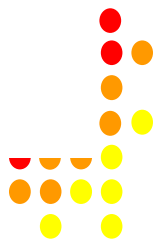
$$\text{And, } t = 2x \Rightarrow \frac{dt}{dx} = 2.$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \frac{2}{\sqrt{1-t^2}} = \frac{2}{\sqrt{1-4x^2}} \quad [\because t = 2x].$$

$$\text{Hence, } \frac{d}{dx} (\sin^{-1}2x) = \frac{2}{\sqrt{1-4x^2}}.$$

(ii) Let $y = \tan^{-1}\sqrt{x}$.

Putting $\sqrt{x} = t$, we get $y = \tan^{-1}t$ and $t = \sqrt{x}$.



$$\text{Now, } y = \tan^{-1}t \Rightarrow \frac{dy}{dt} = \frac{1}{(1+t^2)}.$$

$$\text{And, } t = \sqrt{x} \Rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)} \quad [\because t = \sqrt{x}].$$

$$\text{Hence, } \frac{d}{dx} (\tan^{-1}\sqrt{x}) = \frac{1}{2\sqrt{x}(1+x)}.$$



(iii) Let $y = \cos^{-1}(\cot x)$.

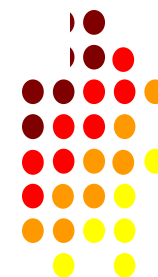
Putting $\cot x = t$, we get $y = \cos^{-1} t$ and $t = \cot x$.

$$\text{Now, } y = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\text{And, } t = \cot x \Rightarrow \frac{dt}{dx} = -\operatorname{cosec}^2 x.$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \frac{\operatorname{cosec}^2 x}{\sqrt{1-t^2}} = \frac{\operatorname{cosec}^2 x}{\sqrt{1-\cot^2 x}} \quad [\because t = \cot x]$$

$$\text{Hence, } \frac{d}{dx} \{ \cos^{-1}(\cot x) \} = \frac{\operatorname{cosec}^2 x}{\sqrt{1-\cot^2 x}}$$



EXAMPLE 2 Differentiate the following w.r.t. x :

(i) $\sec(\tan^{-1}x)$

(ii) $\sin(\tan^{-1}x)$

(iii) $\cot(\cos^{-1}x)$



SOLUTION

(i) Let $y = \sec(\tan^{-1}x)$.

Putting $\tan^{-1}x = t$, we get $y = \sec t$ and $t = \tan^{-1}x$.

$$\text{Now, } y = \sec t \Rightarrow \frac{dy}{dt} = \sec t \tan t.$$

$$\text{And, } t = \tan^{-1}x \Rightarrow \frac{dt}{dx} = \frac{1}{(1+x^2)}.$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \frac{\sec t \tan t}{(1+x^2)} = \frac{(\sqrt{1+\tan^2 t})(\tan t)}{(1+x^2)} \\ &= \frac{(\sqrt{1+x^2})x}{(1+x^2)} = \frac{x}{\sqrt{1+x^2}} \quad [\because t = \tan^{-1}x \Rightarrow \tan t = x] \end{aligned}$$

$$\text{Hence, } \frac{d}{dx} \{\sec(\tan^{-1}x)\} = \frac{x}{\sqrt{1+x^2}}.$$



(ii) Let $y = \sin (\tan^{-1} x)$.

Putting $\tan^{-1} x = t$, we get $y = \sin t$ and $t = \tan^{-1} x$.

Now, $y = \sin t \Rightarrow \frac{dy}{dt} = \cos t$.



$$\text{And, } t = \tan^{-1}x \Rightarrow \frac{dt}{dx} = \frac{1}{(1+x^2)}$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \cos t \cdot \frac{1}{(1+x^2)} = \frac{1}{(1+x^2)^{3/2}}$$

$$\left[\because \tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{1+x^2}} \right]$$

$$\text{Hence, } \frac{d}{dx} \{ \sin (\tan^{-1}x) \} = \frac{1}{(1+x^2)^{3/2}}$$

(iii) Let $y = \cot (\cos^{-1}x)$.

Putting $\cos^{-1}x = t$, we get $y = \cot t$ and $t = \cos^{-1}x$.

$$\text{Now, } y = \cot t \Rightarrow \frac{dy}{dt} = -\operatorname{cosec}^2 t.$$

$$\text{And, } t = \cos^{-1}x \Rightarrow \frac{dt}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \frac{\operatorname{cosec}^2 t}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{3/2}}$$

$$\left[\because \cos t = x \Rightarrow \operatorname{cosec}^2 t = \frac{1}{(1-x^2)} \right]$$

$$\text{Hence, } \frac{d}{dx} \{ \cot (\cos^{-1}x) \} = \frac{1}{(1-x^2)^{3/2}}$$



EXAMPLE 4 Differentiate $\sqrt{\cot^{-1} \sqrt{x}}$ w.r.t. x .



SOLUTION Let $y = \sqrt{\cot^{-1}\sqrt{x}}$.
 Putting $\sqrt{x} = t$ and $\cot^{-1}\sqrt{x} = \cot^{-1}t = u$, we get

$$y = \sqrt{u}, \text{ where } u = \cot^{-1}t \text{ and } t = \sqrt{x}.$$

$$\text{Now, } y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}};$$

$$u = \cot^{-1}t \Rightarrow \frac{du}{dt} = \frac{-1}{(1+t^2)}.$$

$$\text{And, } t = \sqrt{x} \Rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right) = \frac{-1}{4\sqrt{u}(1+t^2)\sqrt{x}}$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}t})(1+t^2)\sqrt{x}}$$

$$[\because u = \cot^{-1}t]$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}\sqrt{x}})(1+x)\sqrt{x}}$$

$$[\because t = \sqrt{x}].$$



EXAMPLE 5 Differentiate $e^{\tan^{-1}\sqrt{x}}$ w.r.t. x .



SOLUTION

$$\text{Let } y = e^{\tan^{-1}\sqrt{x}}.$$

Putting $\sqrt{x} = t$ and $\tan^{-1}\sqrt{x} = \tan^{-1}t = u$, we get

$$y = e^u, \text{ where } u = \tan^{-1}t \text{ and } t = \sqrt{x}.$$

$$\text{Now, } y = e^u \Rightarrow \frac{dy}{du} = e^u;$$

$$u = \tan^{-1}t \Rightarrow \frac{du}{dt} = \frac{1}{(1+t^2)}.$$

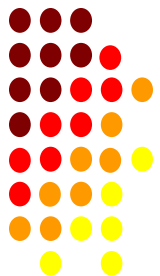
$$\text{And, } t = \sqrt{x} \Rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right) = e^u \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \\ &= e^{\tan^{-1}t} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \quad [\because u = \tan^{-1}t] \end{aligned}$$



$$\begin{aligned}\therefore \frac{dy}{dx} &= \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right) = e^u \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \\ &= e^{\tan^{-1}t} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \quad [\because u = \tan^{-1}t] \\ &= \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)} \quad [\because t = \sqrt{x}].\end{aligned}$$

Hence, $\frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}.$



EXAMPLE ' Show that $\frac{d}{dx} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] = \sqrt{a^2 - x^2}.$



SOLUTION We have

$$\begin{aligned}
 & \frac{d}{dx} \left[\frac{x}{2} \cdot \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \sin^{-1} \frac{x}{a} \right] \\
 &= \frac{d}{dx} \left[\frac{x}{2} \cdot \sqrt{a^2 - x^2} \right] + \frac{a^2}{2} \cdot \frac{d}{dx} \left[\sin^{-1} \frac{x}{a} \right] \\
 &= \frac{x}{2} \cdot \frac{d}{dx} (\sqrt{a^2 - x^2}) + (\sqrt{a^2 - x^2}) \cdot \frac{d}{dx} \left(\frac{x}{2} \right) + \frac{a^2}{2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{1}{a} \\
 &= \frac{x}{2} \cdot \frac{1}{2} (a^2 - x^2)^{-1/2} \cdot (-2x) + (\sqrt{a^2 - x^2}) \cdot \frac{1}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}}
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{x}{2} \cdot \frac{d}{dx}(\sqrt{a^2 - x^2}) + (\sqrt{a^2 - x^2}) \cdot \frac{d}{dx}\left(\frac{x}{2}\right) + \frac{a^2}{2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{1}{a} \\
 &= \frac{x}{2} \cdot \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot (-2x) + (\sqrt{a^2 - x^2}) \cdot \frac{1}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}} \\
 &= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}} \\
 &= \frac{-x^2 + (a^2 - x^2) + a^2}{2\sqrt{a^2 - x^2}} = \frac{(a^2 - x^2)}{\sqrt{(a^2 - x^2)}} = \sqrt{(a^2 - x^2)}.
 \end{aligned}$$

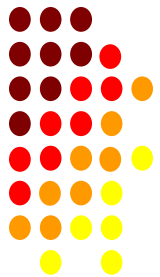
Hence, $\frac{d}{dx} \left[\frac{x}{2} \cdot \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \sin^{-1} \frac{x}{a} \right] = \sqrt{a^2 - x^2}.$



PRACTICE QUESTIONS

Differentiate each of the following w.r.t. x :

1. $\cos^{-1} 2x$
2. $\tan^{-1} x^2$
3. $\sec^{-1} \sqrt{x}$
4. $\sin^{-1} \frac{x}{a}$
5. $\tan^{-1}(\log x)$
6. $\cot^{-1}(e^x)$
7. $\log(\tan^{-1} x)$
8. $\cot^{-1} x^3$
9. $\sin^{-1}(\cos x)$
10. $(1+x^2)\tan^{-1} x$
11. $\tan^{-1}(\cot x)$



ANSWERS

1. $\frac{-2}{\sqrt{1-4x^2}}$

2. $\frac{2x}{(1+x^4)}$

3. $\frac{1}{2x\sqrt{x-1}}$

4. $\frac{1}{\sqrt{a^2-x^2}}$

5. $\frac{1}{x\{1+(\log x)^2\}}$

6. $\frac{-e^x}{(1+e^{2x})}$

7. $\frac{1}{(1+x^2)\tan^{-1}x}$

8. $\frac{-3x^2}{(1+x^6)}$

9. -1

10. $(1+2x\tan^{-1}x)$

11. -1



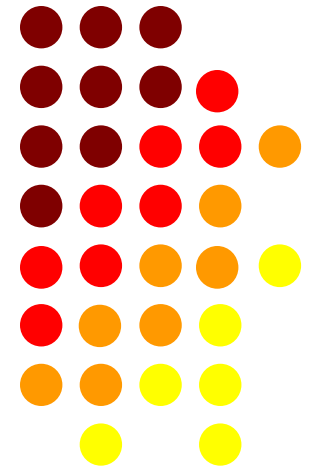
■ ***THANK YOU***



Lecture- 37



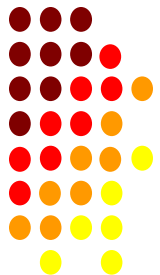
Derivatives of Exponential, Logarithm and Parametric Form



Derivatives of Exponential and Logarithmic Functions

We have (i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(\log x) = \frac{1}{x}$.

$$(ii) \frac{d}{dx}(a^x) = a^x (\log a)$$



EXAMPLE 1 Differentiate each of the following w.r.t. x :

(i) e^{x^2}

(ii) e^{-3x}

(iii) $e^{\cos x}$

..2



EXAMPLE 2 Differentiate each of the following w.r.t. x :

(i) $\sin(\log x)$, $x > 0$

(ii) $\log(\log x)$, $x > 1$



SOLUTION

(i) Let $y = \sin (\log x)$.

Putting $\log x = t$, we get

$$y = \sin t \text{ and } t = \log x$$

$$\Rightarrow \frac{dy}{dt} = \cos t \text{ and } \frac{dt}{dx} = \frac{1}{x}$$



$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \left(\cos t \times \frac{1}{x} \right) = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\text{Hence, } \frac{d}{dx} \{\sin(\log x)\} = \frac{\cos(\log x)}{x}$$

(ii) Let $y = \log(\log x)$.

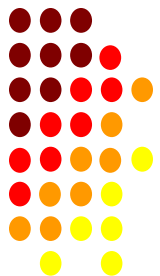
Putting $\log x = t$, we get

$$y = \log t \text{ and } t = \log x$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{t} \text{ and } \frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \left(\frac{1}{t} \times \frac{1}{x} \right) = \left(\frac{1}{\log x} \times \frac{1}{x} \right) = \frac{1}{(x \log x)}$$

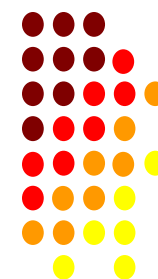
$$\therefore \frac{d}{dx} \{\log(\log x)\} = \frac{1}{(x \log x)}$$



Q.3.

EXAMPLE 3

If $y = e^{\sqrt{\cot x}}$, find $\frac{dy}{dx}$.



SOLUTION Given: $y = e^{\sqrt{\cot x}}$.

Putting $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, we get

$$y = e^u, \quad u = \sqrt{t} \text{ and } t = \cot x$$

$$\Rightarrow \frac{dy}{du} = e^u, \quad \frac{du}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = -\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right)$$

$$= \left\{ e^u \cdot \frac{1}{2\sqrt{t}} \cdot (-\operatorname{cosec}^2 x) \right\} = e^{\sqrt{\cot x}} \cdot \frac{1}{2\sqrt{\cot x}} \cdot (-\operatorname{cosec}^2 x)$$

$$= \frac{(-\operatorname{cosec}^2 x) e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}$$



EXAMPLE 4 If $y = \log \tan \frac{x}{2}$, find $\frac{dy}{dx}$.



SOLUTION Given: $y = \log \tan \frac{x}{2}$.

Putting $\frac{x}{2} = t$ and $\tan \frac{x}{2} = \tan t = u$, we get

$$y = \log u, \quad u = \tan t \text{ and } t = \frac{x}{2}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{u}, \quad \frac{du}{dt} = \sec^2 t \text{ and } \frac{dt}{dx} = \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \right)$$



PRACTICE QUESTIONS

EXERCISE 10B

Differentiate each of the following w.r.t. x:

1. (i) e^{4x} (ii) e^{-5x} (iii) e^{x^3} 2. (i) $e^{\frac{2}{x}}$ (ii) $e^{\sqrt{x}}$ (iii) $e^{-2\sqrt{x}}$

3. (i) $e^{\cot x}$ (ii) $e^{-\sin 2x}$ (iii) $e^{\sqrt{\sin x}}$

4. (i) $\tan(\log x)$ (ii) $\log \sec x$ (iii) $\log \sin \frac{x}{2}$

5. (i) $\log_3 x$ (ii) 2^{-x} (iii) 3^{x+2}

6. (i) $\log \left(x + \frac{1}{x} \right)$ (ii) $\log \sin 3x$ (iii) $\log (x + \sqrt{1+x^2})$ 20



EXAMPLE 1 Differentiate each of the following w.r.t. x :

(i) $\tan^{-1}\left(\frac{1 - \cos x}{\sin x}\right)$

(ii) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$



SOLUTION

(i) Let $y = \tan^{-1}\left(\frac{1 - \cos x}{\sin x}\right) = \tan^{-1}\left\{\frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\}$
 $= \tan^{-1}\left\{\tan \frac{x}{2}\right\} = \frac{x}{2}$.



$$\therefore y = \frac{x}{2}$$

$$\text{Hence, } \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{2} \right) = \frac{1}{2}$$

$$(ii) \text{ Let } y = \tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) = \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right)$$

[on dividing num. and denom. by $\cos x$]

$$= \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} - x \right) \right\} = \left(\frac{\pi}{4} - x \right)$$

$$\therefore y = \left(\frac{\pi}{4} - x \right)$$

$$\text{Hence, } \frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{4} - x \right) = \frac{d}{dx} \left(\frac{\pi}{4} \right) - \frac{d}{dx} (x) = (0 - 1) = -1.$$



EXAMPLE 2 Differentiate w.r.t. x :

(i) $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$

(ii) $\tan^{-1}(\sec x + \tan x)$



SOLUTION (i) Let $y = \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) = \tan^{-1} \left\{ \frac{\sin \left(\frac{\pi}{2} - x \right)}{1 + \cos \left(\frac{\pi}{2} - x \right)} \right\}$

$$= \tan^{-1} \left\{ \frac{2 \sin \left(\frac{\pi}{4} - \frac{x}{2} \right) \cos \left(\frac{\pi}{4} - \frac{x}{2} \right)}{2 \cos^2 \left(\frac{\pi}{4} - \frac{x}{2} \right)} \right\}$$

$$= \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) \right\} = \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

$$\therefore y = \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

$$\text{Hence, } \frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{4} - \frac{x}{2} \right) = \frac{d}{dx} \left(\frac{\pi}{4} \right) - \frac{d}{dx} \left(\frac{x}{2} \right) = \left(0 - \frac{1}{2} \right) = -\frac{1}{2}.$$

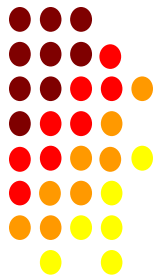


(ii) Let $y = \tan^{-1}(\sec x + \tan x)$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$$

$$= \tan^{-1}\left\{\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right\}$$

$$\left\{\because \cos\left(\frac{\pi}{2} + x\right) = -\sin x; \sin\left(\frac{\pi}{2} + x\right) = \cos x\right\}$$



EXAMPLE 12 Differentiate $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ w.r.t. x .



SOLUTION Let $y = \tan^{-1} \left\{ \frac{\sqrt{1+x^2}-1}{x} \right\}$.

Putting $x = \tan \theta$, we get

$$\begin{aligned} y &= \tan^{-1} \left\{ \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} \right\} = \tan^{-1} \left\{ \frac{\sec\theta-1}{\tan\theta} \right\} \\ &= \tan^{-1} \left\{ \frac{\left(\frac{1}{\cos\theta} - 1 \right)}{\sin\theta} \cdot \cos\theta \right\} = \tan^{-1} \left(\frac{1-\cos\theta}{\sin\theta} \right) \\ &= \tan^{-1} \left\{ \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} \right\} = \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\} \\ &= \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x. \end{aligned}$$



$$\therefore y = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx} (\tan^{-1} x) = \frac{1}{2(1+x^2)}$$



PRACTICE QUESTIONS

Differentiate w.r.t. x:

$$(i) \tan^{-1} \left\{ \frac{\sqrt{1 + \cos x}}{\sqrt{1 - \cos x}} \right\} \quad (ii) \tan^{-1} \left\{ \frac{\sqrt{1 + \sin x}}{\sqrt{1 - \sin x}} \right\}$$

$$\text{If } y = \tan^{-1} \frac{(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})}{(\sqrt{1 + \sin x} - \sqrt{1 - \sin x})}, \text{ find } \frac{dy}{dx}.$$



PRACTICE QUESTIONS

Differentiate w.r.t. x:

(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (ii) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (iii) $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

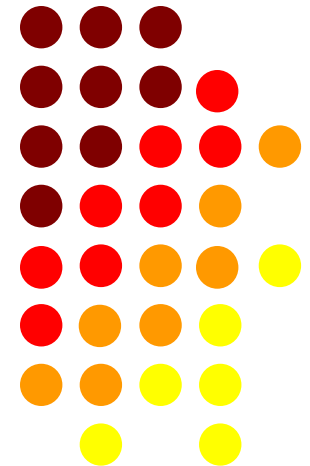


■ ***THANK YOU***



Lecture- 38

Logarithmic Differentiation & Derivative of implicit Function



EXAMPLE : Differentiate x^x w.r.t. x .



SOLUTION Let $y = x^x$.

Taking logarithm on both sides of (i), we get

$$\log y = x \log x.$$

On differentiating both sides of (ii) w.r.t. x , we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{d}{dx} (\log x) + \log x \cdot \frac{d}{dx} (x) \\ &= \left(x \cdot \frac{1}{x} + \log x \cdot 1 \right) = (1 + \log x) \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x).$$



EXAMPLE : Differentiate $(\sin x)^x$ w.r.t. x .



SOLUTION Let $y = (\sin x)^x$.

Taking logarithm on both sides of (i), we get

$$\log y = x \log (\sin x).$$

On differentiating both sides of (ii) w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx} \{\log (\sin x)\} + \log (\sin x) \cdot \frac{d}{dx} (x)$$

$$= x \cdot \frac{1}{\sin x} \cdot \cos x + \log (\sin x) \cdot 1$$

$$= x \cot x + \log (\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y \cdot [x \cot x + \log (\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x [x \cot x + \log (\sin x)].$$



EXAMPLE : Differentiate $(\sin x)^{\log x}$ w.r.t. x .

SOLUTION Let $y = (\sin x)^{\log x}$ (i)

Taking logarithm on both sides of (i), we get

$$\log y = (\log x)(\log \sin x). \quad \dots \text{(ii)}$$

On differentiating both sides of (ii) w.r.t. x , we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= (\log x) \cdot \frac{d}{dx} (\log \sin x) + (\log \sin x) \cdot \frac{d}{dx} (\log x) \\ &= (\log x) \cdot \frac{1}{\sin x} \cdot \cos x + (\log \sin x) \cdot \frac{1}{x} \\ &= (\log x) \cot x + \frac{(\log \sin x)}{x} \\ \Rightarrow \frac{dy}{dx} &= y \cdot \left[(\log x) \cot x + \frac{(\log \sin x)}{x} \right] \\ \Rightarrow \frac{dy}{dx} &= (\sin x)^{\log x} \cdot \left[(\log x) \cot x + \frac{(\log \sin x)}{x} \right]. \end{aligned}$$



EXAMPLE . . If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, find $\frac{dy}{dx}$.



SOLUTION Let $y = u + v$, where $u = (x)^{\cos x}$ and $v = (\cos x)^{\sin x}$.

$$\text{Now, } u = (x)^{\cos x}$$

$$\Rightarrow \log u = (\cos x)(\log x)$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (\cos x) \cdot \frac{d}{dx} (\log x) + (\log x) \cdot \frac{d}{dx} (\cos x)$$

[on differentiating w.r.t. x]

$$= (\cos x) \cdot \frac{1}{x} + (\log x)(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}$$

$$\Rightarrow \frac{du}{dx} = (x)^{\cos x} \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}. \quad \dots (i)$$



$$\text{And, } v = (\cos x)^{\sin x}$$

$$\Rightarrow \log v = (\sin x) \log (\cos x)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = (\sin x) \cdot \frac{d}{dx} \{\log (\cos x)\} + \log (\cos x) \cdot \frac{d}{dx} (\sin x)$$

[on differentiating w.r.t. x]

$$\Rightarrow \frac{dv}{dx} = v \cdot \left\{ (\sin x) \cdot \frac{(-\sin x)}{\cos x} + \log (\cos x) \cdot \cos x \right\}$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}. \quad \dots \text{(ii)}$$

$$\therefore y = (u + v)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (x)^{\cos x} \cdot \left\{ \frac{\cos x}{x} - (\log x) \sin x \right\}$$

$$+ (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}.$$



EXAMPLE : If $x^y = y^x$, find $\frac{dy}{dx}$.



EXAMPLE : If $x^y = y^x$, find $\frac{dy}{dx}$.

SOLUTION Given: $x^y = y^x$

$$\Rightarrow y \log x = x \log y. \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x , we get

$$y \cdot \frac{d}{dx} (\log x) + (\log x) \cdot \frac{d}{dx} (y) = x \cdot \frac{d}{dx} (\log y) + (\log y) \cdot \frac{d}{dx} (x)$$

$$\Rightarrow y \cdot \frac{1}{x} + (\log x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \cdot 1$$

$$\Rightarrow \left(\log x - \frac{x}{y} \right) \frac{dy}{dx} = \left(\log y - \frac{y}{x} \right)$$

$$\Rightarrow \frac{(y \log x - x)}{y} \cdot \frac{dy}{dx} = \frac{(x \log y - y)}{x}$$



$$\Rightarrow \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}$$



EXAMPLE / If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.



SOLUTION We have

$$x^y = e^{x-y} \Rightarrow y \log x = (x - y)$$

$$\Rightarrow (1 + \log x)y = x$$

$$\Rightarrow y = \frac{x}{(1 + \log x)} \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + \log x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2} \\ &= \frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2} = \frac{(1 + \log x - 1)}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2} \end{aligned}$$



Differentiation of Implicit Function



EXAMPLE 1 If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$.



SOLUTION Given: $x^3 + y^3 = 3axy.$

Differentiating both sides of (i) w.r.t. x , we get

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a \cdot \left\{ x \cdot \frac{dy}{dx} + y \cdot 1 \right\}$$

$$\Rightarrow 3(y^2 - ax) \cdot \frac{dy}{dx} = 3(ay - x^2)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{ay - x^2}{y^2 - ax} \right).$$



EXAMPLE 2 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, find $\frac{dy}{dx}$.



SOLUTION Given: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Differentiating both sides of (i) w.r.t. x , we get

$$2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + 2by \cdot \frac{dy}{dx} + 2g + 2f \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow (2ax + 2hy + 2g) + (2hx + 2by + 2f) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{ax + hy + g}{hx + by + f}\right).$$



EXAMPLE 3 If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.



SOLUTION Given: $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$.

Putting $x = \sin \theta$ and $y = \sin \phi$, it becomes

$$\cos \theta + \cos \phi = a(\sin \theta - \sin \phi)$$

$$\Rightarrow \frac{\cos \theta + \cos \phi}{\sin \theta - \sin \phi} = a$$



$$\Rightarrow \frac{2 \cos \left(\frac{\theta + \phi}{2} \right) \cos \left(\frac{\theta - \phi}{2} \right)}{2 \cos \left(\frac{\theta + \phi}{2} \right) \sin \left(\frac{\theta - \phi}{2} \right)} = a$$

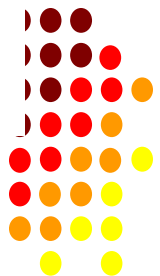
$$\Rightarrow \cot \left(\frac{\theta - \phi}{2} \right) = a \Rightarrow \theta - \phi = 2 \cot^{-1} a$$

$$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2 \cot^{-1} a.$$

On differentiating both sides of (ii) w.r.t. x , we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0.$$

$$\text{Hence, } \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$



EXAMPLE . If $\sin y = x \sin (a + y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$. [CBSE 2012]

SOLUTION $\sin y = x \sin (a + y)$
 $\Rightarrow x = \frac{\sin y}{\sin (a + y)}$... (i)

On differentiating both sides of (i) w.r.t. y , we get

$$\frac{dx}{dy} = \frac{\sin (a + y) \cos y - \sin y \cos (a + y)}{\sin^2(a + y)} \quad [\text{using the quotient rule}]$$
$$= \frac{\sin (a + y - y)}{\sin^2(a + y)} = \frac{\sin a}{\sin^2(a + y)}$$

Hence, $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.



EXAMPLE · If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} = -e^{y-x}$.



SOLUTION Given: $e^x + e^y = e^{x+y}$.

On dividing throughout by e^{x+y} , we get

$$e^{-y} + e^{-x} = 1.$$

On differentiating both sides of (ii) w.r.t. x , we get

$$e^{-y} \cdot \left(\frac{-dy}{dx} \right) + e^{-x}(-1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}.$$



PRACTICE QUESTIONS

Find $\frac{dy}{dx}$, when:

1. $x^2 + y^2 = 4$

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

3. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

4. $x^{2/3} + y^{2/3} = a^{2/3}$

5. $xy = c^2$

6. $x^2 + y^2 - 3xy = 1$

7. $xy^2 - x^2y - 5 = 0$

8. $(x^2 + y^2)^2 = xy$

9. $x^2 + y^2 = \log(xy)$



PRACTICE QUESTIONS

If $y \log x = (x - y)$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

If $\cos y = x \cos (y + a)$, prove that $\frac{dy}{dx} = \frac{\cos^2 (y + a)}{\sin a}$.

EXAMPLE If $x^y + y^x = a^b$, find $\frac{dy}{dx}$.



ANSWERS

$$\begin{array}{ccccc} 1. \frac{-x}{y} & 2. \frac{-b^2x}{a^2y} & 3. -\sqrt{\frac{y}{x}} & 4. \frac{-y^{1/3}}{x^{1/3}} & 5. \frac{-c^2}{x^2} \\ 6. \frac{(2x-3y)}{(3x-2y)} & 7. \frac{(y^2-2xy)}{(x^2-2xy)} & 8. \frac{(y-4xy^2-4x^3)}{(4y^3+4x^2y-x)} & 9. \frac{y(1-2x^2)}{x(2y^2-1)} \end{array}$$



EXAMPLE . If $y = x^{x^{x^{\dots\infty}}}$, prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$.



So, we may write the given function as $y = x^y$.

Now, $y = x^y \Rightarrow \log y = y \log x$.

On differentiating both sides of (i) w.r.t. x , we get

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx} \\ \Rightarrow \left(\frac{1}{y} - \log x \right) \frac{dy}{dx} &= \frac{y}{x} \\ \Rightarrow \frac{(1 - y \log x) \cdot \frac{dy}{dx}}{y} &= \frac{y}{x} \\ \frac{dy}{dx} &= \left\{ \frac{y}{x} \times \frac{y}{(1 - y \log x)} \right\} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}.\end{aligned}$$



EXAMPLE : If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \text{to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{\cos x}{(2y - 1)}$.



SOLUTION We may write the given series as

$$y = \sqrt{\sin x + y} \Rightarrow y^2 = (\sin x + y).$$

On differentiating both sides of (i) w.r.t. x , we get

$$2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y - 1) \cdot \frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(2y - 1)}.$$



EXAMPLE 3 If $y = e^{x+e^{x+e^{x+\dots \text{to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{y}{(1-y)}$.

SOLUTION We may write the given series as

$$y = e^{x+y} \Rightarrow \log y = (x + y). \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

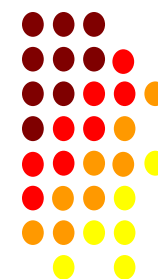
$$\Rightarrow \left(\frac{1}{y} - 1 \right) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{(1-y)}{y} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{(1-y)}$$



EXAMPLE Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.



SOLUTION Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan \theta$, we get

$$u = \sin^{-1}\left(\frac{2\tan \theta}{1+\tan^2 \theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta,$$

$$v = \cos^{-1}\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta.$$

$$\therefore u = v \Rightarrow \frac{du}{dv} = 1.$$



EXAMPLE Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

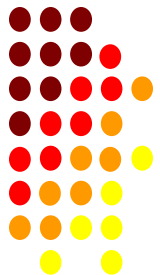
SOLUTION Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Putting $x = \tan \theta$, we get

$$u = \tan^{-1}\left(\frac{2\tan \theta}{1-\tan^2 \theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta,$$

$$v = \sin^{-1}\left(\frac{2\tan \theta}{1+\tan^2 \theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

$$\therefore u = v \Rightarrow \frac{du}{dv} = 1.$$



EXAMPLE Differentiate $\tan^{-1} \left\{ \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right\}$ w.r.t. $\cos^{-1} x^2$

SOLUTION Let $u = \tan^{-1} \left\{ \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right\}$ and $v = \cos^{-1} x^2$.

Then, $\cos^{-1} x^2 = v \Rightarrow x^2 = \cos v$.

Putting $x^2 = \cos v$, we get

$$\begin{aligned} u &= \tan^{-1} \left\{ \frac{\sqrt{1+\cos v} - \sqrt{1-\cos v}}{\sqrt{1+\cos v} + \sqrt{1-\cos v}} \right\} \\ &= \tan^{-1} \left\{ \frac{\sqrt{2\cos^2(v/2)} - \sqrt{2\sin^2(v/2)}}{\sqrt{2\cos^2(v/2)} + \sqrt{2\sin^2(v/2)}} \right\} \end{aligned}$$



$$= \tan^{-1} \left\{ \frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)} \right\} = \tan^{-1} \left\{ \frac{1 - \tan(v/2)}{1 + \tan(v/2)} \right\}$$

[dividing num. and denom. by $\cos(v/2)$]

$$= \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} - \frac{v}{2} \right) \right\} = \left(\frac{\pi}{4} - \frac{v}{2} \right).$$

$$\therefore u = \left(\frac{\pi}{4} - \frac{v}{2} \right) \Rightarrow \frac{du}{dv} = \frac{-1}{2}.$$



1. Derivatives of Parametric Functions

Sometimes x and y are given as functions of a variable t . Then, t is called a *parameter*.

Let $x = f(t)$ and $y = g(t)$. Then,

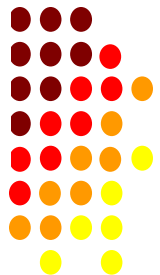
$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = g'(t).$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$

SUMMARY

Let $x = f(t)$ and $y = g(t)$. Then,

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$



EXAMPLE 1 Find $\frac{dy}{dx}$, when $x = a(t + \sin t)$ and $y = a(1 - \cos t)$.

SOLUTION We have:

$$x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t);$$

$$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t.$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx} \right) = \frac{a \sin t}{a(1 + \cos t)} = \frac{2a \sin(t/2) \cos(t/2)}{2a \cos^2(t/2)} = \tan \frac{t}{2}$$



EXAMPLE 2 If $x = a[\cos \theta + \log \tan(\theta/2)]$ and $y = a \sin \theta$, find $\frac{dy}{dx}$ at $\theta = (\pi/4)$.



SOLUTION We have

$$\begin{aligned}x &= a\{\cos \theta + \log \tan (\theta / 2)\} \\ \Rightarrow \frac{dx}{d\theta} &= a\left\{-\sin \theta + \frac{\sec ^2(\theta / 2)}{2 \tan (\theta / 2)}\right\} = a\left\{-\sin \theta + \frac{1}{2 \sin (\theta / 2) \cos (\theta / 2)}\right\} \\ &= a\left\{-\sin \theta + \frac{1}{\sin \theta}\right\} = \frac{a(1 - \sin ^2 \theta)}{\sin \theta} = \frac{a \cos ^2 \theta}{\sin \theta}.\end{aligned}$$

$$\text{And, } y = a \sin \theta \Rightarrow \frac{dy}{d\theta} = a \cos \theta.$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{d\theta} \times \frac{d\theta}{dx}\right) = \left(a \cos \theta \cdot \frac{\sin \theta}{a \cos ^2 \theta}\right) = \tan \theta.$$

$$\therefore \left[\frac{dy}{dx}\right]_{\theta = \pi / 4} = \tan \frac{\pi}{4} = 1.$$



■ ***THANK YOU***

