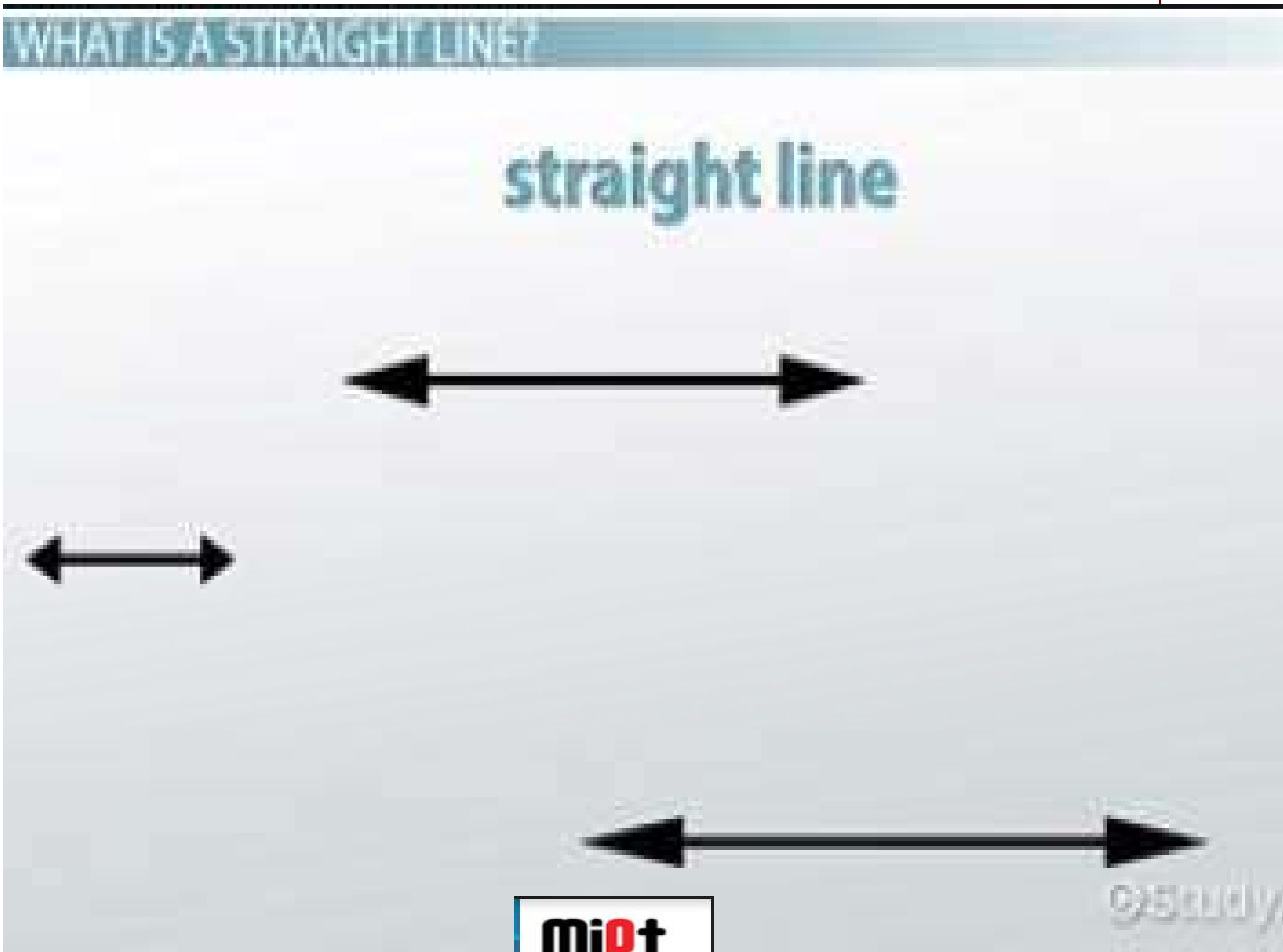
UNIT-3

Lecture- 17

Introduction of Straight line & Slope of a line


Straight Lines


WHAT IS A STRAIGHT LINE?
straight line

Straight Lines

The shortest distance between two points is called straight line.

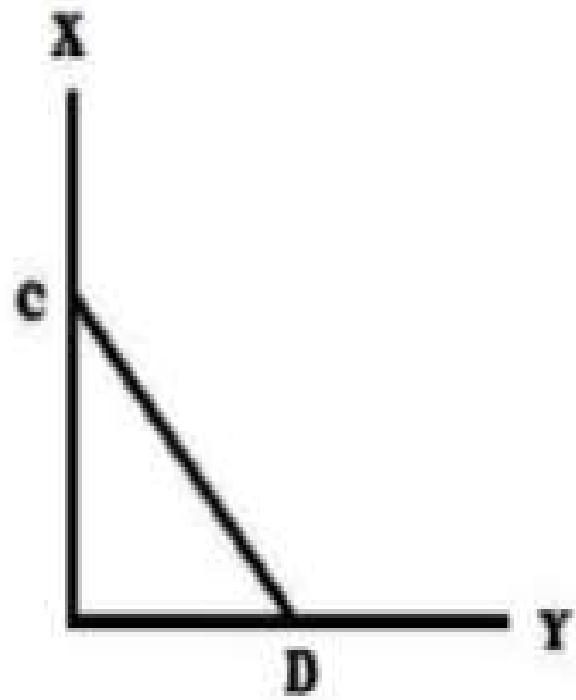
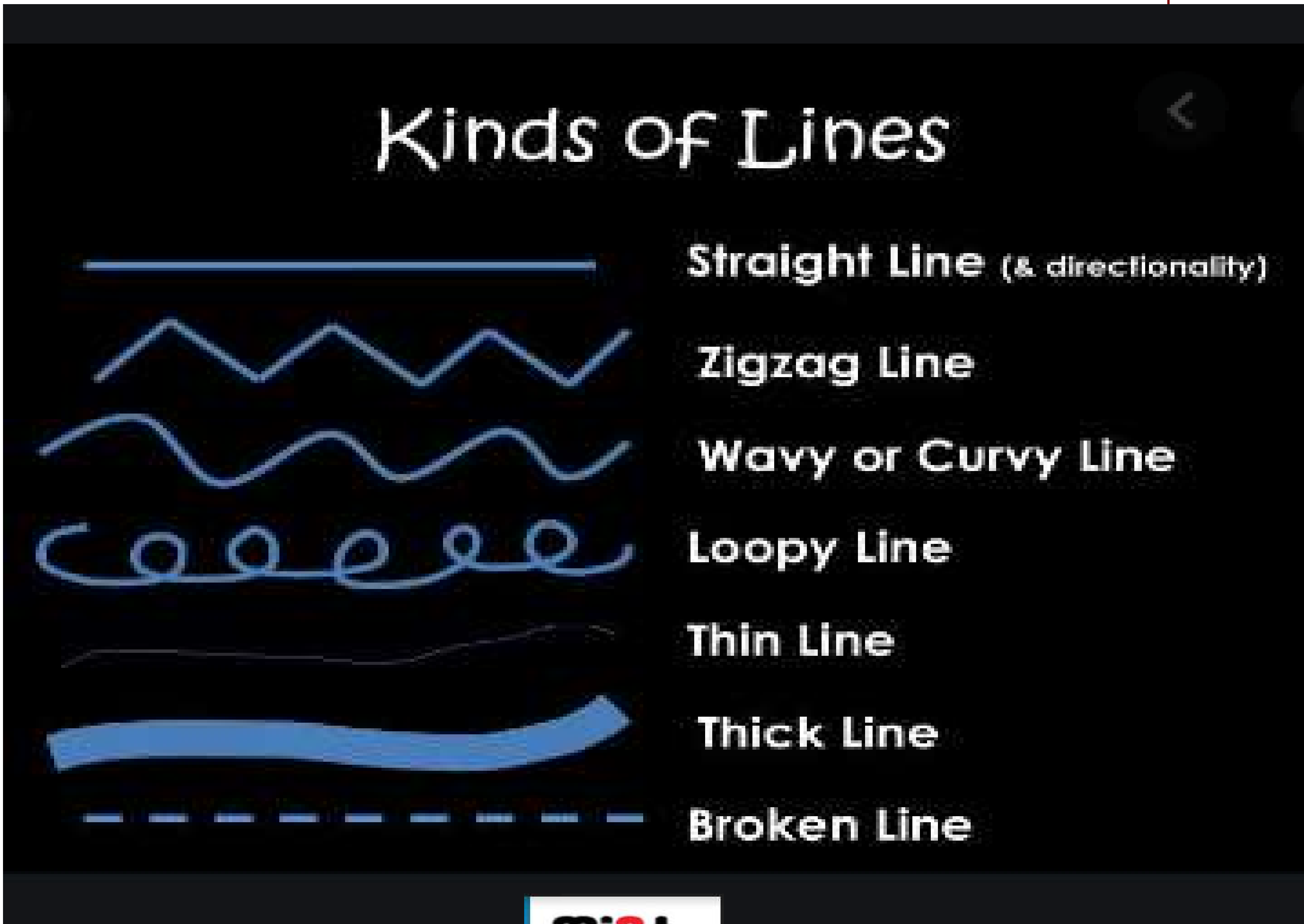


Figure: Straight Line


Kinds of Lines
Straight Line (& directionality)
Zigzag Line
Wavy or Curvy Line
Loopy Line
Thin Line
Thick Line
Broken Line

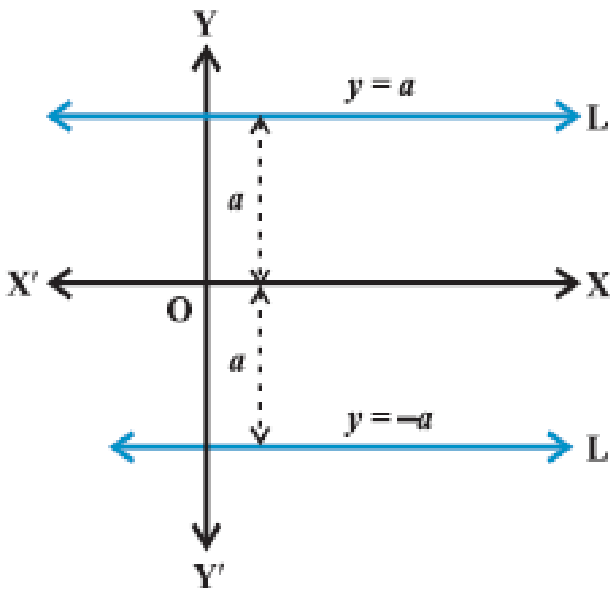
DIFFERENT KINDS OF A LINE

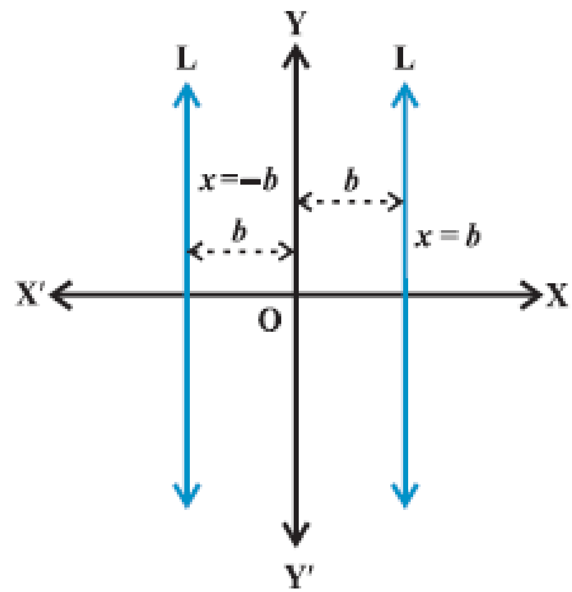
HORIZONTAL LINE *Any line parallel to the x-axis or the x-axis itself is called a horizontal line.*

VERTICAL LINE *Any line parallel to the y-axis or the y-axis itself is called a vertical line.*

OBLIQUE LINE *A line which is neither horizontal nor vertical is called a oblique line.*

Horizontal and vertical lines





If a horizontal line *L* is at a distance a from the *x*-axis then ordinate of every point lying on the line is either *a* or – *a* Therefore, equation of the line *L* is either $y = a$ or $y = -a$.

Similarly, the equation of a vertical line at a distance b from the *y*-axis is either $x = b$ or $x = -b$

Parallel and Perpendicular Lines

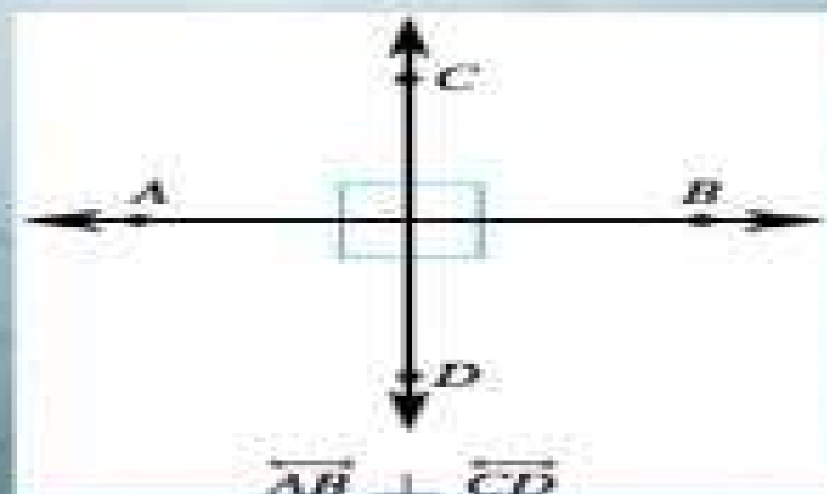
NOTE: PARALLEL LINES NEVER INTERSECT TO EACH OTHER

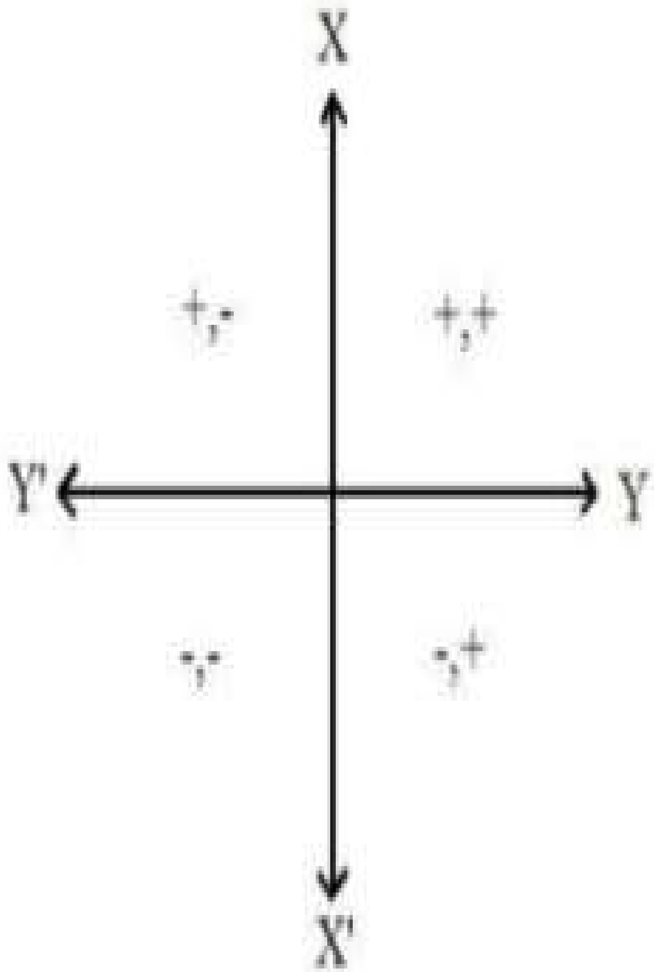
What is a perpendicular line?

perpendicular lines

Two lines that intersect to form four right angles

Example:





- Two mutually perpendicular lines intersect each other and divide the plane into four parts
- Each part is called Quadrant
- The intersecting point, is known as the origin
- The lines are called rectangular axis

KEY POINTS

- 1. SLOPE OF A LINE
- 2. SLOPE OF A LINE WHEN COORDINATES OF 2 POINTS ARE GIVEN.
- 3.ANGLE BETWEEN TWO LINES
- 4.CONDITION OF PARALLEL & PERPENDICULAR LINES
- 5.COLLINEARITY OF THREE POINTS

1. Distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1 *Find the distance between the points (2, –3) and (–6, 3).*

SOLUTION Let $A(2, -3)$ and $B(-6, 3)$ be the given points. Then,

$$AB = \sqrt{(-6-2)^2 + \{3-(-3)\}^2} = \sqrt{(-8)^2 + (3+3)^2} = \sqrt{(-8)^2 + 6^2}$$

$$= \sqrt{64+36} = \sqrt{100} = 10 \text{ units.}$$

EXAMPLE 2 *Using the distance formula, prove that the points $A(-2, 3)$, $B(1, 2)$ and $C(7, 0)$ are collinear.*

SOLUTION We have

$$AB = \sqrt{(1+2)^2 + (2-3)^2} = \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ units;}$$

$$BC = \sqrt{(7-1)^2 + (0-2)^2} = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10} \text{ units;}$$

$$AC = \sqrt{(7+2)^2 + (0-3)^2} = \sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10} \text{ units.}$$

$$\therefore \quad AB + BC = (\sqrt{10} + 2\sqrt{10}) \text{ units} = 3\sqrt{10} \text{ units} = AC.$$

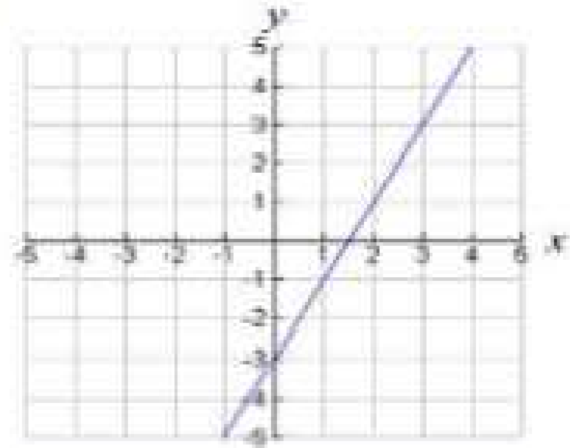
Thus, $AB + BC = AC$, showing that the points A, B, C are collinear.

1. Find the distance between the points:

(i) $A(2, -3)$ and $B(-6, 3)$ (ii) $C(-1, -1)$ and $D(8, 11)$

SLOPE OF A STRAIGHT LINE

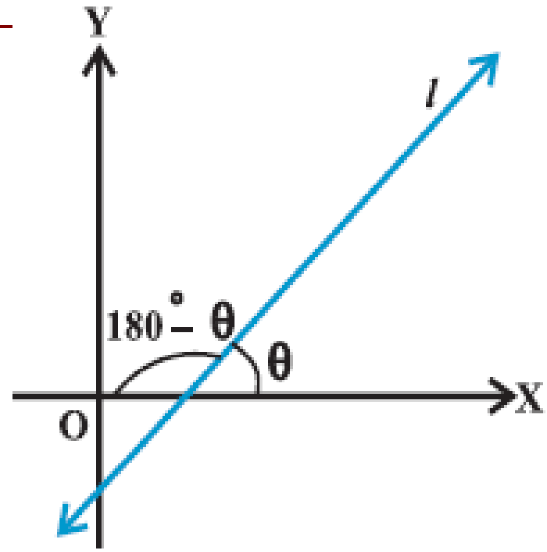
- One of the most important properties of a straight line is in how it angles away from the horizontal. The concept is reflected in slope of the line.



STRAIGHT LINE

Slope of a Line

If θ is the angle made by a line with positive direction of *x-axis in* ***anticlockwise direction***, then the value of **tan** θ is called the **slope of the line** and is denoted by m.



Note: The slope of a line whose inclination is 90° is not defined.

Thus, $m = \tan\theta,\ \theta \neq 90°$

EXAMPLE 1 *Find the slope of a line whose inclination is*

(i) 45°   *(ii)* 60°   *(iii)* 150°

SOLUTION Let m be the slope of the line. Then,

(i) $m = \tan 45° = 1.$

(ii) $m = \tan 60° = \sqrt{3}.$

(iii) $m = \tan 150° = \tan (180° - 30°) = -\tan 30° = \frac{-1}{\sqrt{3}}.$

REMARK 1 *The slope of a horizontal line is 0.*

We know that the inclination of a horizontal line is 0°.

So, slope of a horizontal line is $m = \tan 0° = 0.$

REMARK 2 *The slope of a vertical line is not defined.*

We know that the inclination of vertical line is 90°.

So, slope of a vertical line is $m = \tan 90°$, which is not defined.

EXAMPLE 2 *What is the inclination of a line whose slope is*

(i) zero? *(ii) positive?* *(iii) negative?* *(iv) not defined?*

SOLUTION Let θ be the inclination of the given line. Then, $m = \tan\theta$.

(i) $m = 0 \Rightarrow \tan\theta = 0$

$\Rightarrow \theta = 0^\circ$ $[\because\ 0^\circ \le \theta < 180^\circ]$

(ii) $m > 0 \Rightarrow \tan\theta > 0$

$\Rightarrow$ θ lies between 0° and 90°

$\Rightarrow$ θ is acute.

(iii) $m < 0 \Rightarrow \tan\theta < 0$

$\Rightarrow$ θ lies between 90° and 180°

$\Rightarrow$ θ is obtuse.

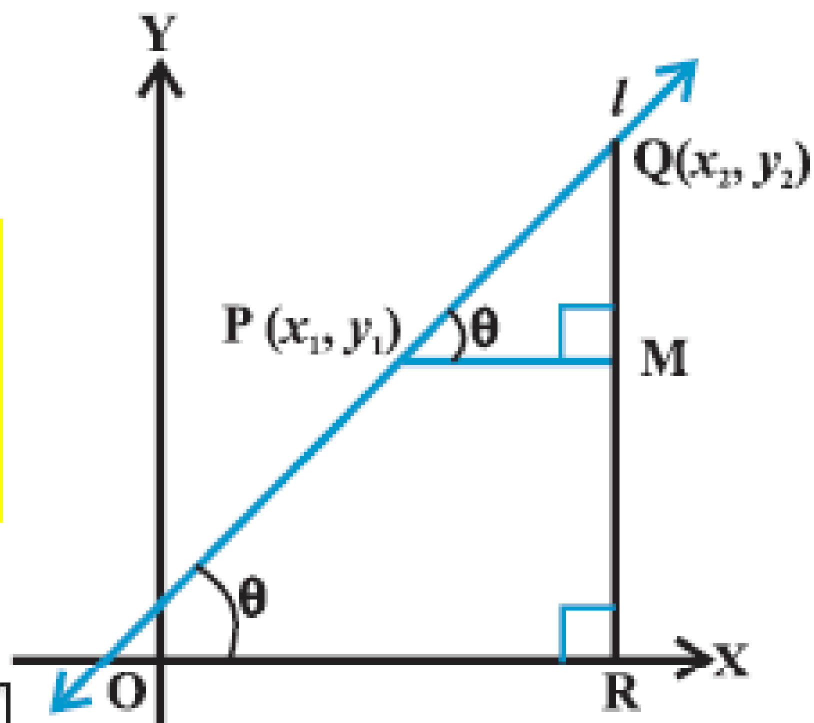
(iv) We know that a vertical line is the only line whose slope is not defined. And, the inclination of a vertical line is 90°.

Hence, the inclination of a line whose slope is not defined, is 90°.

Slope of a line when coordinates of any two points on the line are given

The ***slope*** of a line passing through points **P(x_1, y_1)** and **Q (x_2, y_2)** is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$



EXAMPLE 3 *Find the slope of the line passing through the points*

(i) (–2, 3) *and* (8, –5) *(ii)* (4, – 3) *and* (6, – 3) *(iii)* (3, – 1) *and* (3, 2)

SOLUTION (i) Let $A(-2, 3)$ and $B(8, -5)$ be the given points. Then,

$$\text{slope of } AB = \frac{-5-3}{8-(-2)} = \frac{-8}{10} = \frac{-4}{5}. \qquad \left[\because \; m = \frac{(y_2 - y_1)}{(x_2 - x_1)}\right]$$

(ii) Let $C(4, -3)$ and $D(6, -3)$ be the given points. Then,

$$\text{slope of } CD = \frac{-3-(-3)}{6-4} = \frac{-3+3}{2} = \frac{0}{2} = 0.$$

ALITER The points $C(4, -3)$ and $D(6, -3)$ have the same y-coordinate. So, CD is a line parallel to the x-axis. Hence, its slope is 0.

(iii) Let $P(3, -1)$ and $Q(3, 2)$ be the given points. Then,

$$\text{slope of } PQ = \frac{2-(-1)}{3-3} = \frac{3}{0}, \text{ which is not defined.}$$

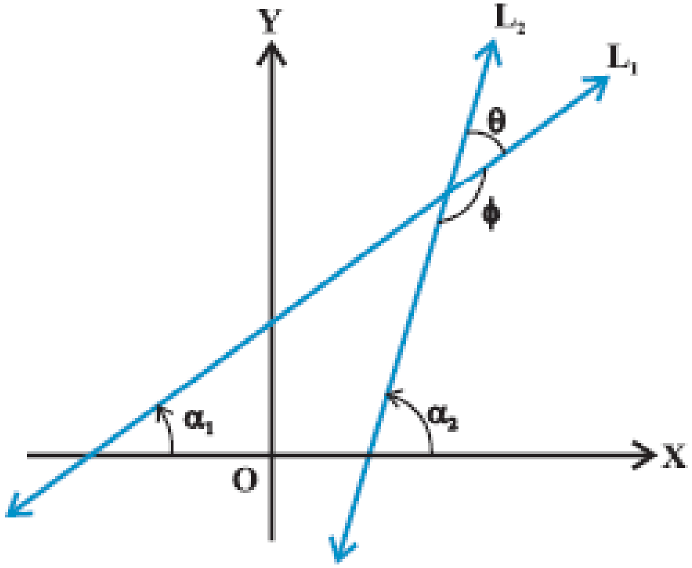
Lecture- 18

Angle Between two Lines

Angle between two lines

The angle θ between the two lines having slopes m_1 and m_2 is given by

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$



1. If A (-2, 1), B (2, 3) and C (-2, -4) are three points, fine the angle between the straight lines AB and BC.

Solution:

Let the slope of the line AB and BC are m_1 and m_2 respectively.

Then,

$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = ½$ and

$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$

Let θ be the angle between AB and BC. Then,

$\tan\theta = |\frac{m_2-m_1}{1+m_1m_2}| = |\frac{\frac{7}{4}-\frac{1}{2}}{1+\frac{7}{4}\cdot\frac{1}{2}}| = |\frac{\frac{10}{8}}{\frac{15}{8}}| = \pm\frac{2}{3}$.

$\Rightarrow \theta = \tan^{-1}(\frac{2}{3})$, which is the required angle.

SUMMARY

Let L_1 and L_2 be two lines whose slopes are m_1 and m_2 respectively. Then,

(i) $L_1 \parallel L_2 \Leftrightarrow m_1 = m_2$

(ii) $L_1 \perp L_2 \Rightarrow m_1 m_2 = -1$

EXAMPLE 7 *Show that the line joining the points (2, −3) and (−5, 1) is parallel to the line joining the points (7, −1) and (0, 3).*

SOLUTION Let $A(2, -3)$, $B(-5, 1)$, $C(7, -1)$ and $D(0, 3)$ be the given points.

Then, slope of $AB = \dfrac{1-(-3)}{-5-2} = \dfrac{4}{-7} = -\dfrac{4}{7}$

And, slope of $CD = \dfrac{3-(-1)}{0-7} = \dfrac{4}{-7} = -\dfrac{4}{7}$.

$\therefore$ slope of AB = slope of CD.

Hence, $AB \parallel CD$.

EXAMPLE 8 *Show that the line joining the points (2, –5) and (–2, 5) is perpendicular to the line joining the points (6, 3) and (1, 1).*

SOLUTION Let $A(2, -5)$, $B(-2, 5)$, $C(6, 3)$ and $D(1, 1)$ be the given points.

Let m_1 and m_2 be the slopes of AB and CD respectively. Then,

$$m_1 = \text{slope of } AB = \frac{5-(-5)}{-2-2} = \frac{10}{-4} = -\frac{5}{2}.$$

$$m_2 = \text{slope of } CD = \frac{1-3}{1-6} = \frac{-2}{-5} = \frac{2}{5}.$$

$$\therefore \quad m_1 m_2 = \left(\frac{-5}{2}\right) \times \frac{2}{5} = -1.$$

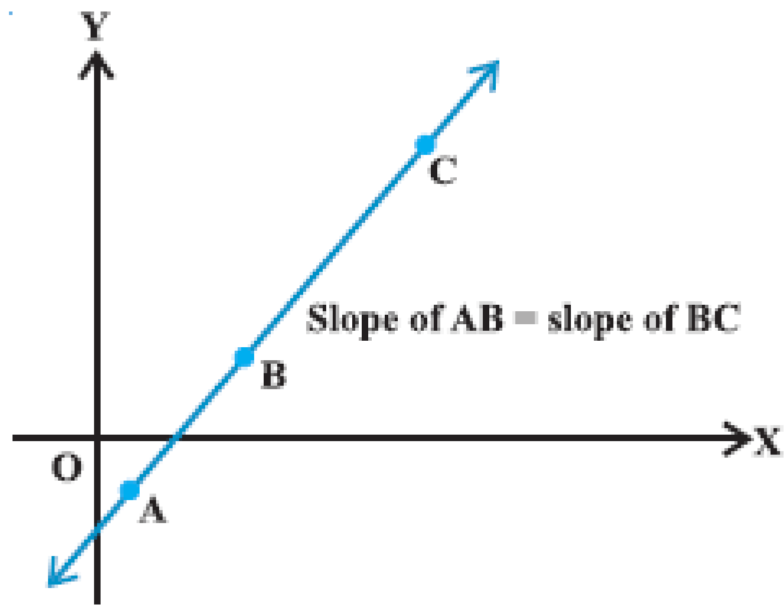
Hence, $AB \perp CD$.

Collinearity of three points

If three points **A** (h, k), **B** (x_1, y_1) and **C**(x_2,y_2) are such that slope of AB = slope of BC, *i.e.*,

$$\frac{y_1-k}{x_1-h}=\frac{y_2-y_1}{x_2-x_1}$$

$$or,\ (h-x_1)(y_2-y_1)=(k-y_1)(x_2-x_1)$$



then they are said to be **collinear.**

EXAMPLE *Using slopes, show that the points (5, 1), (1, −1) and (11, 4) are collinear.*

SOLUTION Let $A(5, 1)$, $B(1, -1)$ and $C(11, 4)$ be the given points. Then,

$$\text{slope of } AB = \frac{(-1-1)}{(1-5)} = \frac{-2}{-4} = \frac{1}{2}$$

$$\text{and slope of } BC = \frac{4-(-1)}{11-1} = \frac{5}{10} = \frac{1}{2}.$$

$\therefore$ slope of AB = slope of BC

$\Rightarrow$ $AB \parallel BC$ and have a point B in common

$\Rightarrow$ A, B, C are collinear.

Hence, the given points are collinear.

Lecture- 19

Various forms of the equation of a line

Various forms of the equation of a line

- 1. GENERAL FORM
- 2.POINT SLOPE FORM
- 3. TWO POINT FORM
- 4.INTERCEPT FORM

- GENERAL FORM

The "General Form" of the equation of a straight line is:

$$y = mx + b$$

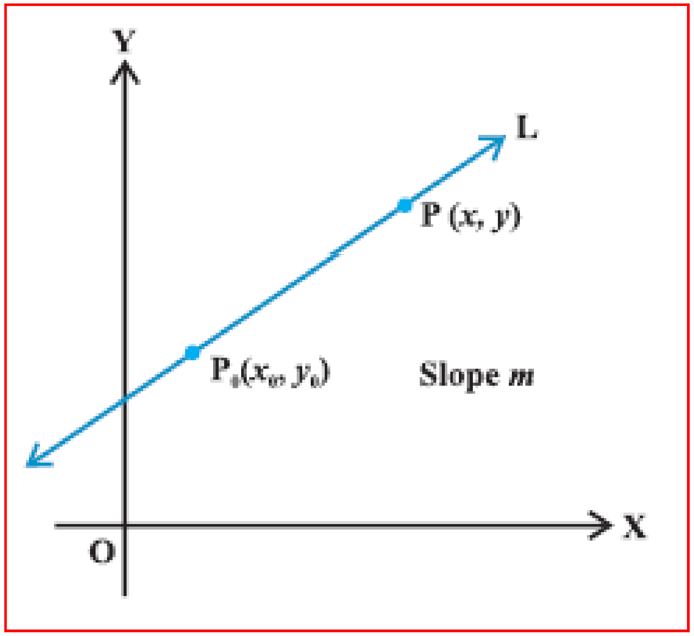
General equation of a line

Any equation of the form $\boldsymbol{Ax + By + C = 0}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are simultaneously not zero, is called the **general equation** of a line.

Point-slope form

The equation of a line having slope ***m*** and passing through the point **(x_0, y_0)** is given by

$$y - y_0 = m(x - x_0)$$



EXAMPLE 1 *Find the equation of a line passing through the point (4, 3) and having slope 2.*

SOLUTION We know that the equation of a line with slope m and passing through the point (x_1, y_1) is given by

$$(y - y_1) = m(x - x_1).$$

Here, $m = 2$, $x_1 = 4$ and $y_1 = 3$.

Hence, the required equation is

$$(y - 3) = 2(x - 4), \quad \text{i.e.,} \quad 2x - y - 5 = 0.$$

EXAMPLE 2 *Find the equation of a line which makes an angle of 135° with the x-axis and passes through the point (3, 5).*

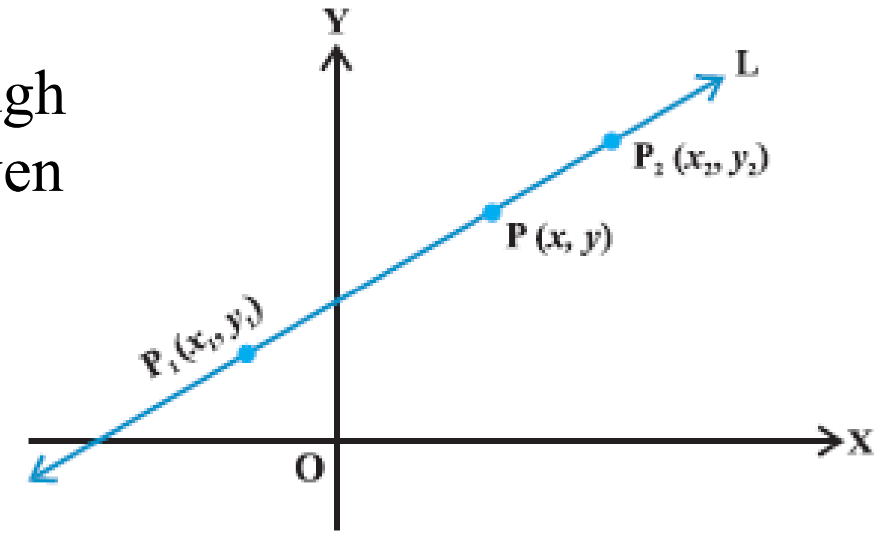
SOLUTION Here, $m = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1.$

Hence, the required equation is

$$\frac{y-5}{x-3} = -1 \Leftrightarrow (y-5) = 3 - x \Leftrightarrow x + y - 8 = 0.$$

Two-point form

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

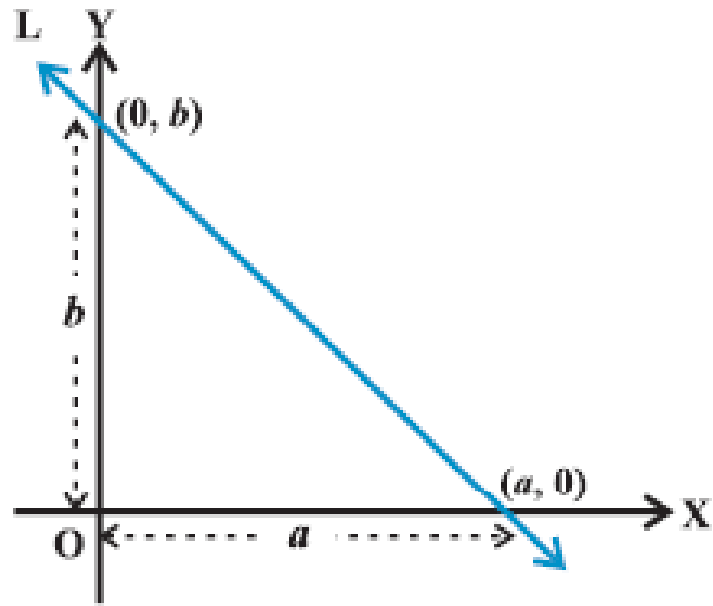


$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Intercept - form

The equation of the line making intercepts ***a*** and ***b*** on *x*- and y-axis respectively is given by

$$\frac{x}{a}+\frac{y}{b}=1$$



1. Find the equation of a straight line whose slope = -7 and which intersects the y-axis at a distance of 2 units from the origin.

Solution:

Here $m = -7$ and $b = 2$. Therefore, the equation of the straight line is $y = mx + b \Rightarrow y = -7x + 2 \Rightarrow 7x + y - 2 = 0$.

2. Find the slope and y-intercept of the straight-line $4x - 7y + 1 = 0$.

Solution:

The equation of the given straight line is

$4x - 7y + 1 = 0$

$\Rightarrow 7y = 4x + 1$

$\Rightarrow y = 4/7x + 1/7$

Now, compare the above equation with the equation $y = mx + b$ we get,

$$m = 4/7 \text{ and } b = 1/7.$$

Therefore, the slope of the given straight line is 4/7 and its y-intercept = 1/7 units.

Lecture- 21

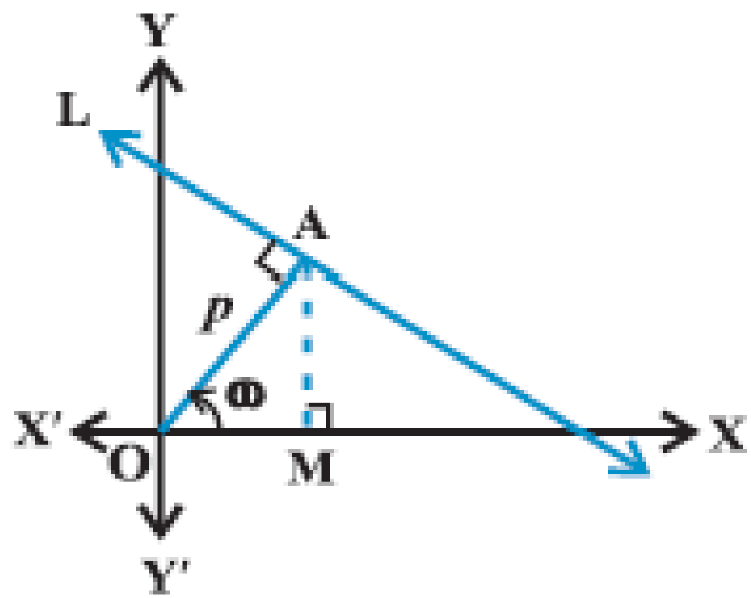
Normal form of the equation of a line

Normal form

Suppose a non-vertical line is known to us with following data:

(a) Length of the perpendicular (normal) p from origin to the line.

(b) Angle ω which normal makes with the positive direction of *x-axis*.



Then the equation of such a line is given by

$$x \cos \omega + y \sin \omega = p$$

Find the equation of the straight line which is at a of distance 7 units from the origin and the perpendicular from the origin to the line makes an angle 45° with the positive direction of x-axis.

Solution:

We know that the equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x-axis is x cos α + y sin α = p.

Here p = 7 and α = 45°

Therefore, the equation of the straight line in normal form is

x cos 45° + y sin 45° = 7

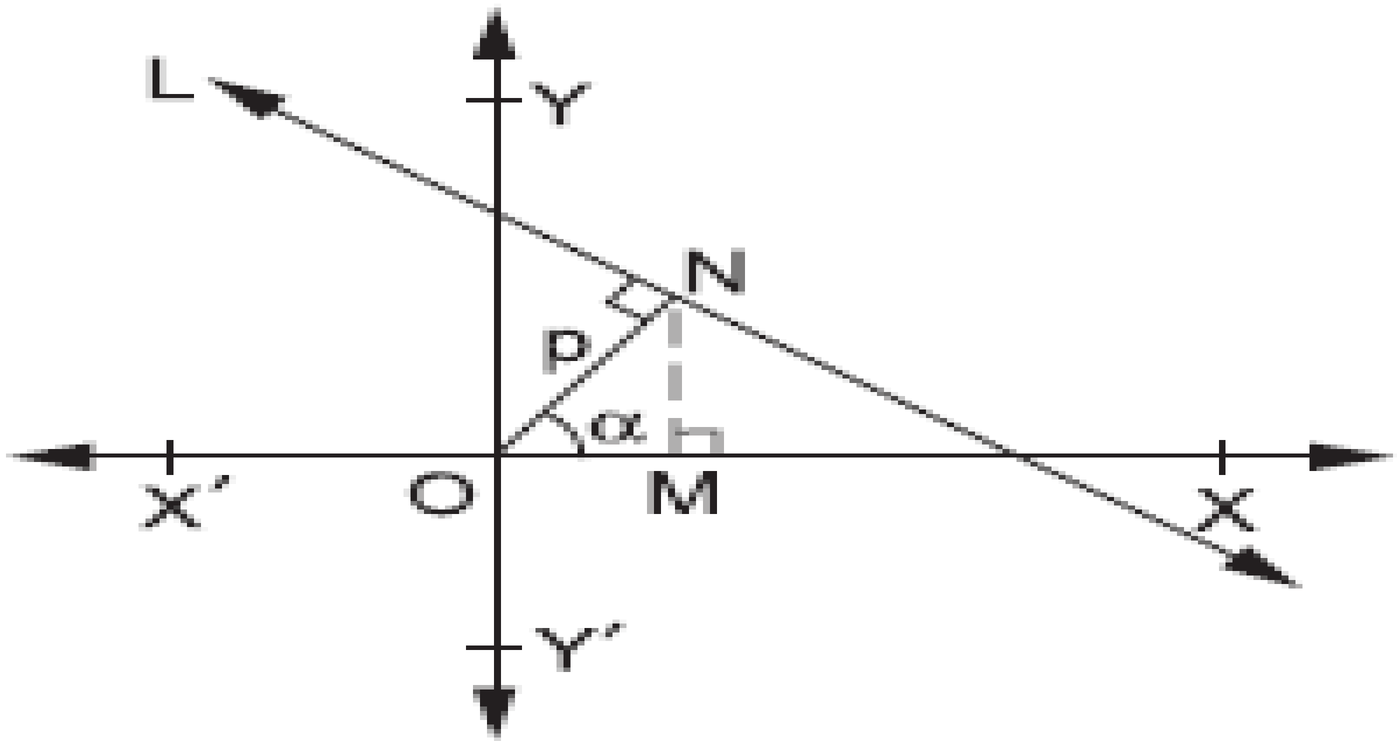
$\Rightarrow x \cdot \frac{1}{\sqrt{2}} + y \cdot \frac{1}{\sqrt{2}} = 7$

$\Rightarrow \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 7$

⇒ x + y = 7√2, which is the required equation.

EQUATION OF A LINE IN NORMAL FORM

THEOREM 1 *Let p be the length of perpendicular (or normal) from the origin to a given non-vertical line L, and let α be the angle between the normal and the positive direction of the x-axis. Then, prove that the equation of the line L is given by*

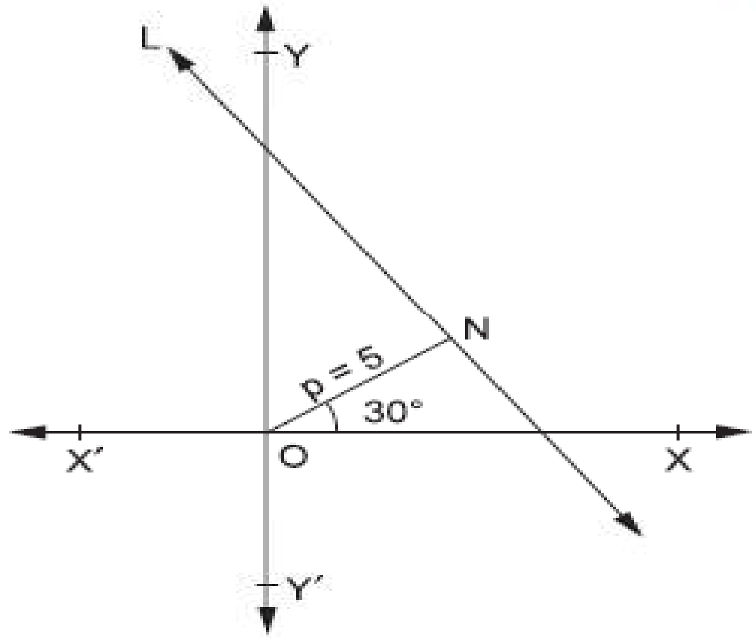


EXAMPLE 1 *Find the equation of a line whose perpendicular distance from the origin is 5 units and the angle between the positive direction of the x-axis and the perpendicular is 30°.*

SOLUTION Here $p = 5$ units and $\alpha = 30°$.

So, the equation of the given line in normal form is

$$x \cos\alpha + y \sin\alpha = p, \text{ where } \alpha = 30° \text{ and } p = 5 \text{ units}$$



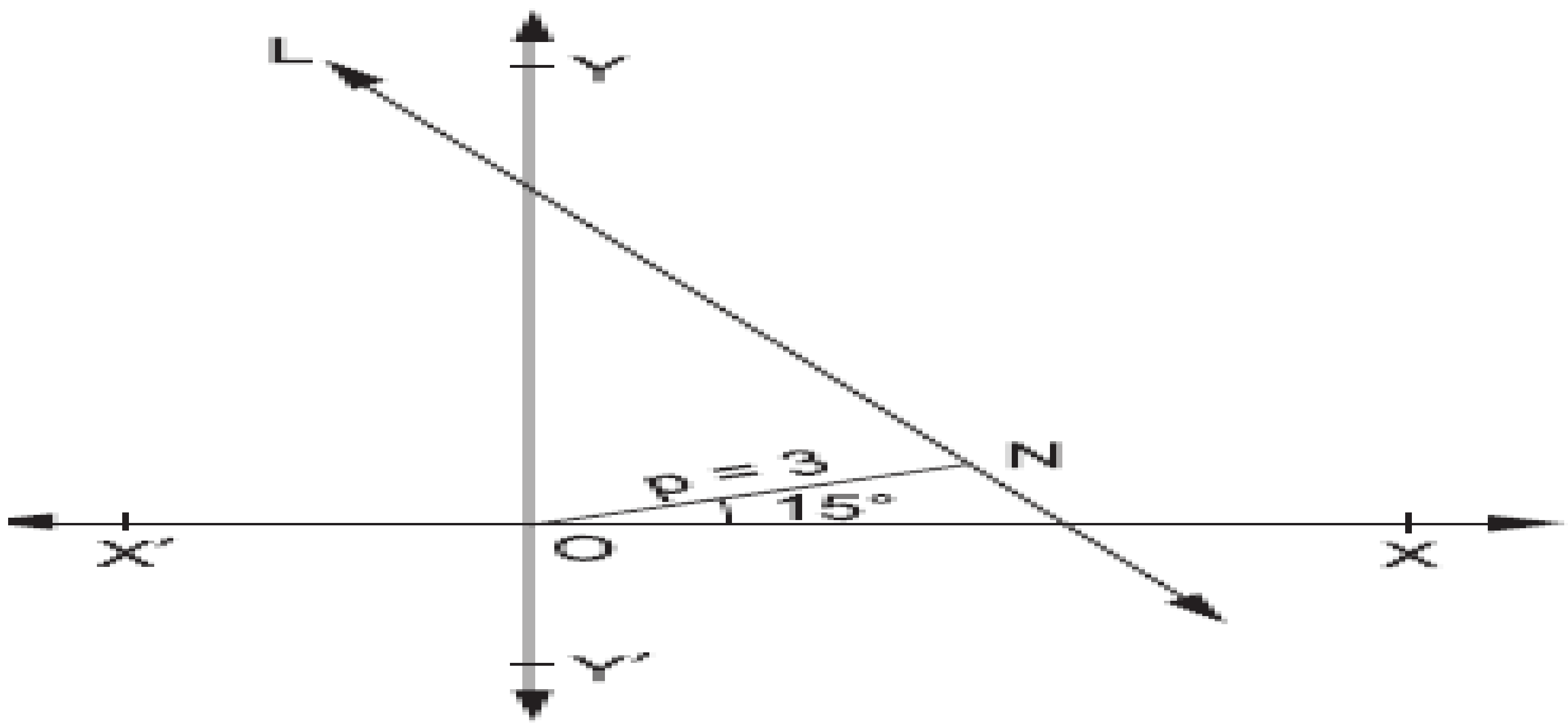
$\Leftrightarrow \quad x \cos 30° + y \sin 30° = 5$

$$\Leftrightarrow \quad x\left(\frac{\sqrt{3}}{2}\right)+y\left(\frac{1}{2}\right)=5$$

$$\Leftrightarrow \quad \sqrt{3}x+y-10=0,$$

which is the required equation.

EXAMPLE 2 *Find the equation of the line whose perpendicular distance from the origin is 3 units and the angle between the positive direction of x-axis and the perpendicular is 15°.*

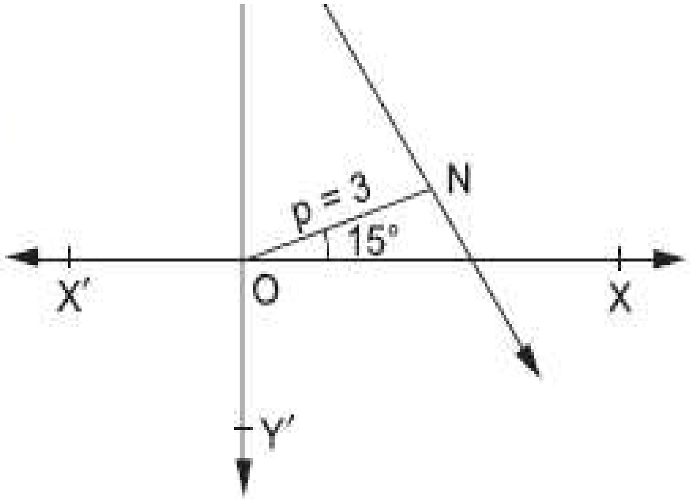


L
Y
N
p = 3
15°
X'
O
X
Y'

SOLUTION Here $p = 3$ units and $\alpha = 15°$.

So, the equation of the given line in normal form is

$x \cos\alpha + y \sin\alpha = p$, where $\alpha = 15°$ and $p = 3$ units



$$\Leftrightarrow \quad x\cos 15° + y \sin 15° = 3$$

$$\Leftrightarrow \quad \frac{x(\sqrt{3}+1)}{2\sqrt{2}} + \frac{y(\sqrt{3}-1)}{2\sqrt{2}} = 3$$

[$\because$ $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$]

$\Leftrightarrow \quad (\sqrt{3}+1)x + (\sqrt{3}-1)y - 6\sqrt{2} = 0$, which is the required equation.

EXAMPLE 3 *Find the equation of a line whose perpendicular distance from the origin is $\sqrt{8}$ units and the angle between the positive direction of the x-axis and the perpendicular is 135°.*

Different forms of $Ax + By + C = 0$

The general form of the line can be reduced to various forms as given below:

- Slope intercept form
- Intercept form
- Normal Form

Slope intercept form

If B ≠ 0, then **A*x* + *By* + *C* = 0** can be written as

$$y = \frac{-A}{B}x + \frac{-C}{B}$$

or $y = mx + c$, where $m = \frac{-A}{B}$ and $c = \frac{-C}{B}$

If $B = 0$, then $x = \frac{-C}{A}$

which is a vertical line whose slope is not defined and *x-intercept*

Is $\frac{-C}{A}$

Intercept form

If $C \neq 0$, then $Ax + By + C = 0$ can be written as

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1$$

OR

$$\frac{x}{a} + \frac{y}{b} = 1$$

Where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$

If $C = 0$, then $Ax + By + C = 0$ can be written as $Ax + By = 0$ which is a line passing through the **origin** and therefore has **zero intercepts** on the axes.

Normal Form

The normal form of the equation $Ax + By + C = 0$ is $\boldsymbol{x}$ $\mathbf{cos}\ \boldsymbol{\omega} + \boldsymbol{y}\ \mathbf{sin}\ \boldsymbol{\omega} = \boldsymbol{p}$ where,

$$\cos\omega = \pm\frac{A}{\sqrt{A^2 + B^2}}, \sin\omega = \pm\frac{B}{\sqrt{A^2 + B^2}}$$

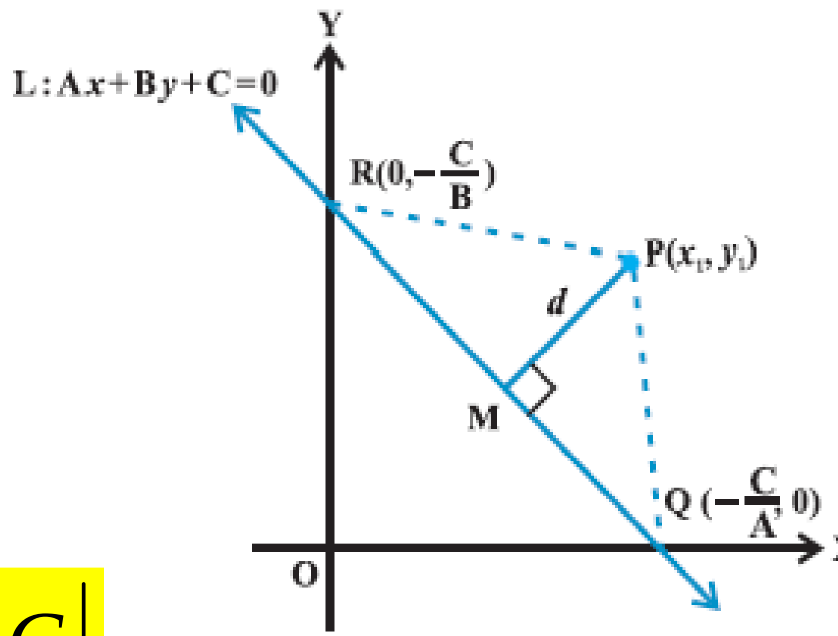
and

$$p = \pm\frac{C}{\sqrt{A^2 + B^2}}$$

Distance of a point from a line

The perpendicular distance ***d*** of a point ***P*** (x_1, y_1) from the line $Ax + By + C = 0$ is given by

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$



1. Find the perpendicular distance between the line 4x - y = 5 and the point (2, - 1).

Solution:

The equation of the given straight line is 4x - y = 5

or, 4x - y - 5 = 0

If *Z* be the perpendicular distance of the straight line from the point (2, - 1), then

$$Z = \frac{|4 \cdot 2 - (-1) - 5|}{\sqrt{4^2 + (-1)^2}}$$

$$= \frac{|8+1-5|}{\sqrt{16+1}}$$

$$= \frac{|4|}{\sqrt{17}}$$

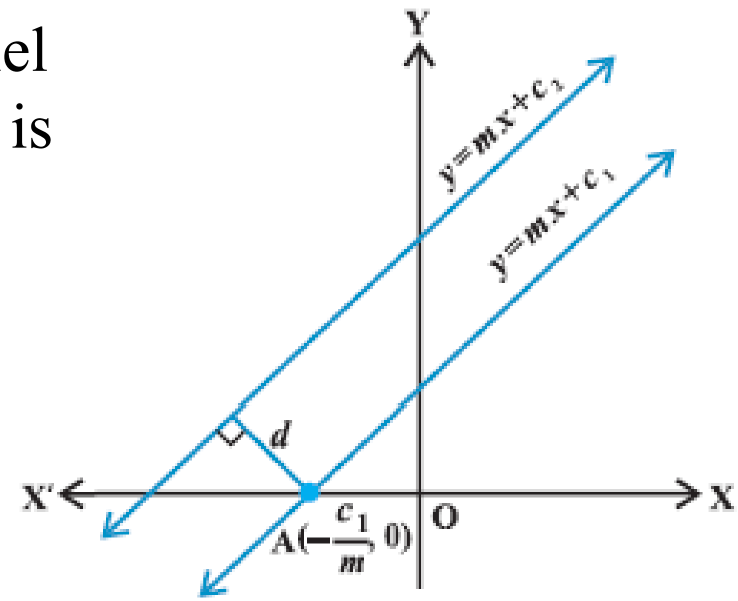
$$= \frac{4}{\sqrt{17}}$$

Therefore, the required perpendicular distance between the line 4x - y = 5 and the point (2, - 1)= $\frac{4}{\sqrt{17}}$ units.

Distance between two parallel lines

The distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \left| \frac{C_1 - C_2}{\sqrt{1+m^2}} \right|$$



1. Find the equation of the straight line which is parallel to $5x - 7y = 0$ and passing through the point $(2, - 3)$.

Solution:

The equation of any straight line parallel to the line $5x - 7y = 0$ is $5x - 7y + \lambda = 0$ .... (i) [Where λ is an arbitrary constant].

If the line (i) passes through the point $(2, - 3)$ then we shall have,

$5 \cdot 2 - 7 \cdot (-3) + \lambda = 0$

$\Rightarrow 10 + 21 + \lambda = 0$

$\Rightarrow 31 + \lambda = 0$

$\Rightarrow \lambda = -31$

Therefore, the equation of the required straight line is $5x - 7y - 31 = 0$.

Intersection of two given lines

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) intersecting if
$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

(ii) parallel and distinct if
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

(iii) coincident if
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

1. Find the equation of a straight line parallel to x-axis at a distance of 10 units above the x-axis.

Solution:

We know that the equation of a straight line parallel to x-axis at a distance b from it is $y = b$.

Therefore, the equation of a straight line parallel to x-axis at a distance 10 units above the x-axis is $y = 10$.

2. Find the equation of a straight line parallel to y-axis at a distance of 20 units on the right hand side of y-axis.

Solution:

We know that the equation of a straight line is parallel and to the right of x-axis at a distance a, then its equation is $x = a$.

Therefore, the equation of a straight line parallel to y-axis at a distance of 20 units on the right hand side of y-axis is $x = 20$

Some more examples:

1. Find the angle which the straight line perpendicular to the straight lin $\sqrt{3}x + y = 1$, makes with the positive direction of the x-axis.

Solution:

The given equation of the straight line $\sqrt{3}x + y = 1$

Covert the above equation into slope-intercept form we get,

$y = -\sqrt{3}x + 1$........................ (i)

Let us assume that the given straight line (i) makes an angle θ with the positive direction of the x-axis.

Then, the slope of the straight line (i) will be tan θ

Hence, we must have, tan = $-\sqrt{3}$ [Since, the slope of the straight line $y = -\sqrt{3}x + 1$ is $-\sqrt{3}$]

$\Rightarrow \tan\theta = -\tan 60° = \tan(180° - 60°) = \tan 120°$

$\Rightarrow \tan\theta = 120°$

Since the straight line (i) makes an angle 120° with the positive direction of the x-axis, hence a straight line perpendicular to the line (i) will make an angle 120° - 90° = 30° with the positive direction of the x-axis.

2. Prove that P (4, 3), Q (6, 4), R (5, 6) and S (3, 5) are the angular poir of a square.

Solution:

We have,

PQ = $\sqrt{(6-4)^2+(4-3)^2}$ = √5

QR = $\sqrt{(6-4)^2+(5-4)^2}$ = √5

RS = $\sqrt{(5-6)^2+(3-5)^2}$ = √5 and

SP = $\sqrt{(5-3)^2+(3-4)^2}$ = √5

Therefore, PQ = QR = RS = SP.

Now, m_1 = Slope of PQ = $\frac{4-3}{6-4}$ = ½

m_2 = Slope of QR = $\frac{6-4}{5-6}$ = -2 and

m_3 = Slope of RS = $\frac{5-6}{3-5}$ = ½

Clearly, $m_1 \cdot m_2$ = ½ · (-2) = -1 and $m_1 = m_3$.

This shows that PQ is perpendicular to QR and PQ is parallel to RS.

Thus, PQ = QR = RS = SP, PQ ⊥ QR and PQ is parallel to RS.

Thence, PQRS is a square.

HOME WORK

In Exercises , find the equation of the line which satisfy the given conditions:

1. Write the equations for the *x*-and *y*-axes.
2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.
3. Passing through (0, 0) with slope *m*.
4. Passing through $\left(2, 2\sqrt{3}\right)$ and inclined with the *x*-axis at an angle of 75°.
5. Intersecting the *x*-axis at a distance of 3 units to the left of origin with slope –2.
6. Intersecting the *y*-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the *x*-axis.
7. Passing through the points (–1, 1) and (2, – 4).

. The equation

$$6x^2 + xy - 12y^2 - 13x + 6y + 6 = 0$$

represents

(a) a pair of straight lines through the origin
(b) a pair of perpendicular straight lines
(c) a pair of parallel straight lines
(d) a pair of straight lines not passing through the origin, neither parallel nor perpendicular

Solution

(d) We have,

$$6x^2 + xy - 12y^2 - 13x + 6y + 6 = 0$$

Here, $a = 6,\ h = \frac{1}{2},\ b = -12,$

$$g = -\frac{13}{2}$$

$$f = 3,\ c = 6$$

$\because \quad a + b = 6 - 12 = -6 \neq 0$

$\therefore$ A pair of line is not perpendicular

and $\because h^2 \neq ab$

$\therefore$ Pair of line is not parallel

Hence, pair of straight line is not passing through origin, neither parallel nor perpendicular.

Q -2

If the point (α, α) lies between the lines $|2x + y| = 5$ then select one of the most appropriate option

(a) $|\alpha| < \frac{5}{3}$

(b) $|\alpha| < \frac{7}{2}$

(c) $|\alpha| < \frac{11}{3}$

(d) $|\alpha| < \frac{5}{2}$

SOLUTION-2

(a) Given lines are

$$|2x + y| = 5$$

i.e. $$2x + y = 5 \quad \ldots(i)$$

or $$2x + y = -5 \quad \ldots(ii)$$

Point (α, α) lie on the line

$$y = x \quad \ldots(iii)$$

$\because$ Line $y = x$ intersect the line Eqs. (i) and (ii) at points $\left(\frac{5}{3}, \frac{5}{3}\right)$ and $\left(\frac{-5}{3}, \frac{-5}{3}\right)$.

[Here, we find point of intersects by solving line Eqs. (i), (iii) and lines Eqs. (ii) and (iii)]

$\therefore$ α lie in between $\frac{-5}{3}$ and $\frac{5}{3}$

i.e. $$\frac{-5}{3} < \alpha < \frac{5}{3}$$

$\therefore$ $$|\alpha| < \frac{5}{3}$$

The points of the curve $y = x^3 + x - 2$ at which its tangents are parallel to the straight line $y = 4x - 1$ are :

(A) $(1, 0), (-1, -4)$

(B) $(2, 7), (-2, -11)$

(C) $(0, -2), \left(2^{\frac{1}{3}}, 2^{\frac{1}{3}}\right)$

(D) $\left(-2^{\frac{1}{3}}, -2^{\frac{1}{3}}\right), (0, -4)$

The points of the curve $y = x^3 + x - 2$ at which its tangents are parallel to the straight line $y = 4x - 1$ are :

(A) $(1, 0), (-1, -4)$

(B) $(2, 7), (-2, -11)$

(C) $(0, -2), \left(2^{\frac{1}{3}}, 2^{\frac{1}{3}}\right)$

(D) $\left(-2^{\frac{1}{3}}, -2^{\frac{1}{3}}\right), (0, -4)$

Solution: (a)

Given,

$$y = x^3 + x - 2 \quad \ldots(i)$$

$$y = 4x - 1 \quad \ldots(ii)$$

Slope of tangent to the curve (i)

$$\frac{dy}{dx} = 3x^2 + 1$$

Slope of tangent at point (α, β) is

$$\left.\frac{dy}{dx}\right|_{(\alpha, \beta)} = 3\alpha^2 + 1 \quad \ldots(iii)$$

Given, tangent of cu[illegible]parallel to line (ii).

$\therefore$ Slope of line (ii) is 4.

$\therefore$ From Eq. (iii), we get

$$3\alpha^2 + 1 = 4$$

$$\Rightarrow \quad \alpha = \pm 1$$

$\therefore$ (α, β) lie on curve (i).

$$\therefore \quad \beta = (\pm 1)^3 + (\pm 1) - 2$$

$$\Rightarrow \quad \beta = 0, -4$$

$\therefore$ Points are (1, 0) and (–1, –4)

- THANK YOU

Lecture- 20 &21

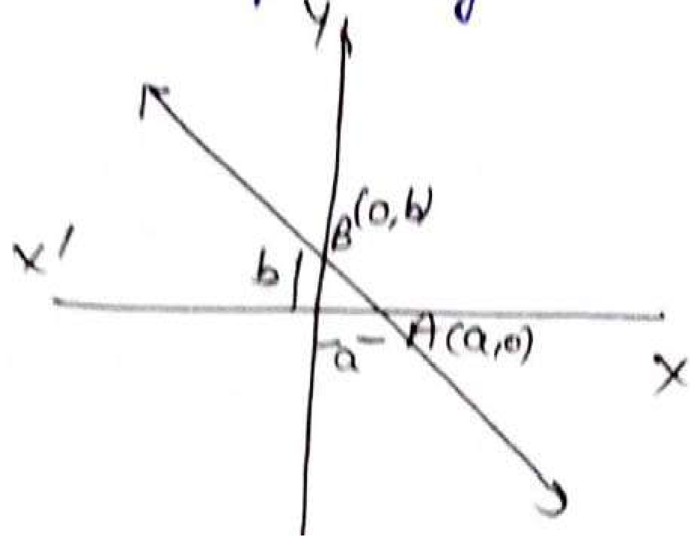
Equations of a line intercepts form and normal form

Equation of a line in intercepts form :

[2018-19]

The equation of a line making intercepts a & b on the x-axis and the y-axis respectively

$$\frac{x}{a} + \frac{y}{b} = 1$$



Q-1 Find the equation of the straight line which passes through the point (3, 4) and the intercept made by this line on y-axis is two times the intercept on x-axis.

[2013-14]

Sol. The equation of straight line is

$$\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow (1)$$

Given as $b = 2a \longrightarrow (2)$

And the equation (1) passes through the point (3, 4) so by (1)

$$\frac{3}{a} + \frac{4}{2a} = 1$$

$$\frac{6+4}{2a} = 1 \Rightarrow \frac{10}{2a} = 1$$

$$2a = 10, \quad a = 5$$

$b = 2 \times 5 = 10$

By (1) the required equation is -

$$\boxed{\frac{x}{5} + \frac{y}{10} = 1}$$ Ans.

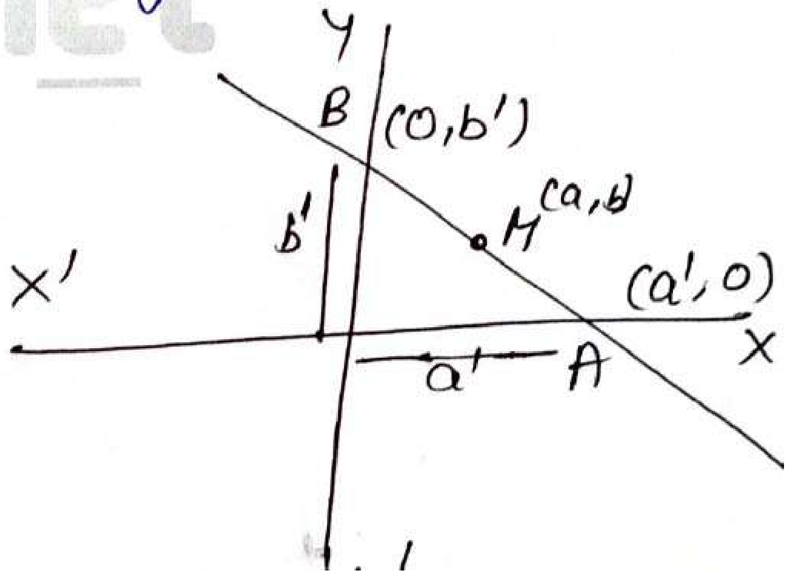
Q-2 If M(a,b) is the mid point of a line segment intercepted between the axes, show that the equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 2$$

Sol. Let the required equation be

$$\frac{x}{a'} + \frac{y}{b'} = 1 \longrightarrow (1)$$

then x-intercept $= a'$, y-intercept $= b'$



So we have,

$$a = \frac{0 + a'}{2}$$

$$\Rightarrow a' = 2a$$

$$\& \; b = \frac{0 + b'}{2}$$

$\Rightarrow b' = 2b$

Use in ①

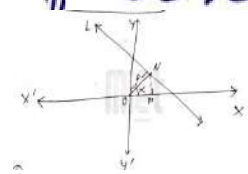
$$\frac{x}{2a} + \frac{y}{2b} = 1$$

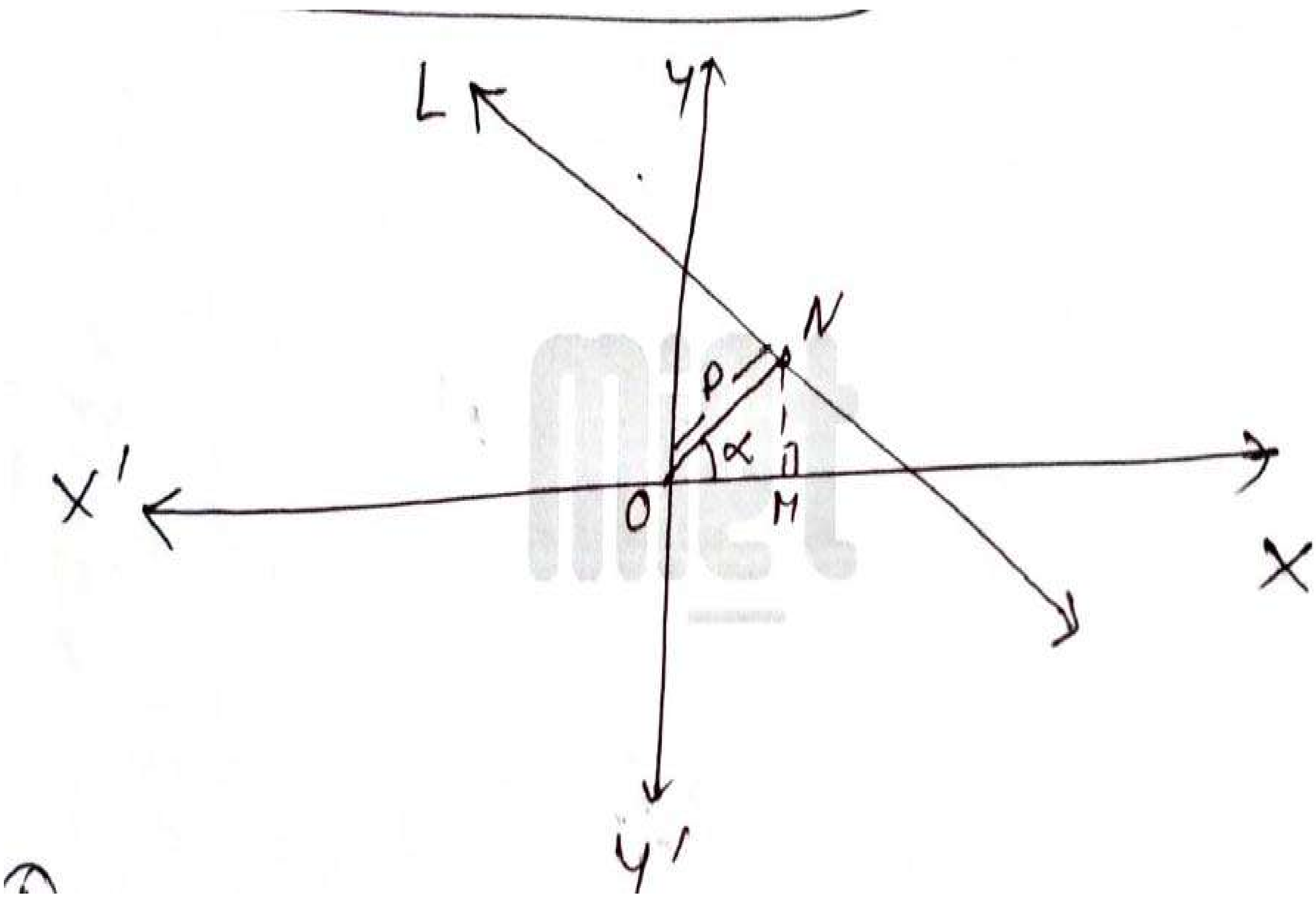
$$\boxed{\frac{x}{a} + \frac{y}{b} = 2}$$ Ans

Equation of a line in normal form:-

Let p be the length of perpendicular (or normal) from the origin to a given non-vertical line L and let α be the angle between the normal and the positive direction of the x-axis then the equation of line is.

$$\boxed{x\cos\alpha + y\sin\alpha = p}$$




L
Y
N
P
α
X'
O
M
X
Y'

(1) Find the equation of a line whose perpendicular distance from the origin is 5 units and the angle between the positive direction of the x-axis and the perpendicular is 30°.

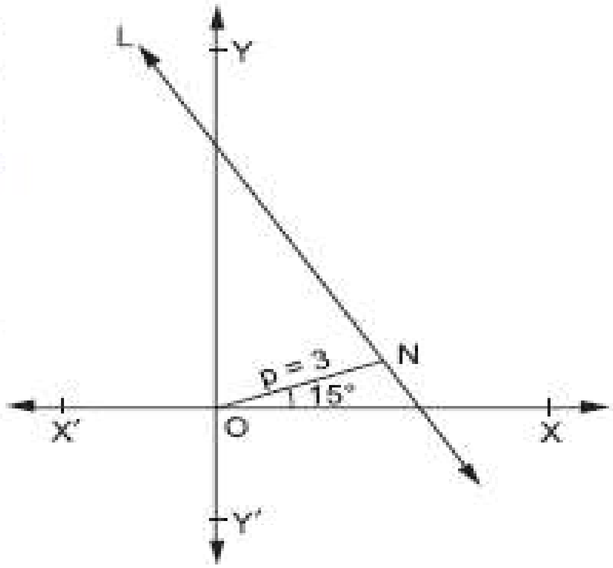
Sol. $p = 5$, $\alpha = 30^\circ$

So the required equation is -

$$x\cos 30^\circ + y\sin 30^\circ = 5 \Rightarrow \frac{x\sqrt{3}}{2} + y \times \frac{1}{(2)} = 5$$

$$\boxed{x\sqrt{3} + y = 10}$$ Ans.

EXAMPLE 2 *Find the equation of the line whose perpendicular distance from the origin is 3 units and the angle between the positive direction of x-axis and the perpendicular is 15°.*



SOLUTION Here $p = 3$ units and $\alpha = 15°$.

So, the equation of the given line in normal form is

$x\cos\alpha + y\sin\alpha = p$, where $\alpha = 15°$ and $p = 3$ units

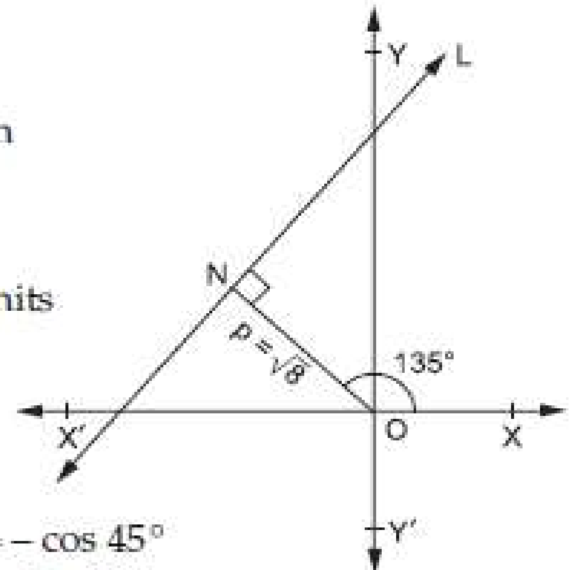
$\Leftrightarrow \quad x\cos 15° + y\sin 15° = 3$

$$\Leftrightarrow \quad \frac{x(\sqrt{3}+1)}{2\sqrt{2}} + \frac{y(\sqrt{3}-1)}{2\sqrt{2}} = 3$$

[$\because$ $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$
$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$]

$\Leftrightarrow \quad (\sqrt{3}+1)x + (\sqrt{3}-1)y - 6\sqrt{2} = 0$, which is the required equation.

EXAMPLE 3 *Find the equation of a line whose perpendicular distance from the origin is $\sqrt{8}$ units and the angle between the positive direction of the x-axis and the perpendicular is 135°.*



SOLUTION Here $p = \sqrt{8}$ units and $\alpha = 135°$.

So, the equation of the given line in normal form is

$$x \cos \alpha + y \sin \alpha = p,$$

where $\alpha = 135°$ and $p = \sqrt{8}$ units

$$\Leftrightarrow \quad x \cos 135° + y \sin 135° = \sqrt{8}$$

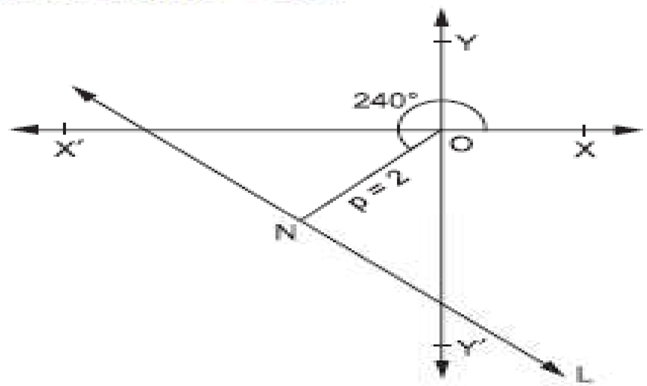
$$\Leftrightarrow \quad x\left(\frac{-1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) = \sqrt{18}$$

[$\because$ $\cos 135° = \cos (180° - 45°) = -\cos 45°$
$\sin 135° = \sin (180° - 45°) = \sin 45°$]

$$\Leftrightarrow \quad -x + y = 4 \quad \Leftrightarrow \quad x - y + 4 = 0,$$ which is the required equation.

EXAMPLE 4 *Find the equation of a line whose perpendicular distance from the origin is 2 units and the angle between the perpendicular segment and the positive direction of the x-axis is 240°.*

SOLUTION Here, $p = 2$ units and $\alpha = 240°$.



So, the equation of the given line in normal form is

$x \cos\alpha + y \sin\alpha = p$, where $\alpha = 240°$ and $p = 2$ units

$\Leftrightarrow \quad x \cos 240° + y \sin 240° = 2$

$\Leftrightarrow \quad x\left(\frac{-1}{2}\right) + y\left(\frac{-\sqrt{3}}{2}\right) = 2$ $[\because \cos 240° = \cos(180° + 60°) = -\cos 60°$

$\sin 240° = \sin(180° + 60°) = -\sin 60°]$

$\Leftrightarrow \quad -x - \sqrt{3}y = 4 \Leftrightarrow x + \sqrt{3}y + 4 = 0$, which is the required equation.

PRACTICE QUESTIONS

1. Find the equation of the line for which

   (i) $p = 3$ and $\alpha = 45°$ (ii) $p = 5$ and $\alpha = 135°$

   (iii) $p = 8$ and $\alpha = 150°$ (iv) $p = 3$ and $\alpha = 225°$

   (v) $p = 2$ and $\alpha = 300°$ (vi) $p = 4$ and $\alpha = 180°$

2. The length of the perpendicular segment from the origin to a line is 2 units and the inclination of this perpendicular is α such that $\sin\alpha = \frac{1}{3}$ and α is acute. Find the equation of the line.

3. Find the equation of the line which is at a distance of 3 units from the origin such that $\tan\alpha = \frac{5}{12}$, where α is the acute angle which this perpendicular makes with the positive direction of the x-axis.

ANSWERS

1. (i) $x + y - 3\sqrt{2} = 0$ (ii) $x - y + 5\sqrt{2} = 0$ (iii) $\sqrt{3}x - y + 16 = 0$

   (iv) $x + y + 3\sqrt{2} = 0$ (v) $x - \sqrt{3}y - 4 = 0$ (vi) $x + 4 = 0$

2. $2\sqrt{2}x + y - 6 = 0$

3. $12x + 5y - 39 = 0$

- ***THANK YOU***

Lecture- 22

Two point form of the equation of a line

Hence, its slope is not defined.

EXAMPLE . *If the slope of the line passing through the points* (2, 5) *and* (x, 3) *is* 2, *find the value of* x.

SOLUTION Let $A(2, 5)$ and $B(x, 3)$ be the given points. Then,

$$\text{slope of } AB = \frac{3-5}{x-2} = \frac{-2}{(x-2)}.$$

$$\therefore \quad \frac{-2}{(x-2)} = 2 \Leftrightarrow 2x - 4 = -2$$

$$\Leftrightarrow 2x = 2 \Leftrightarrow x = 1.$$

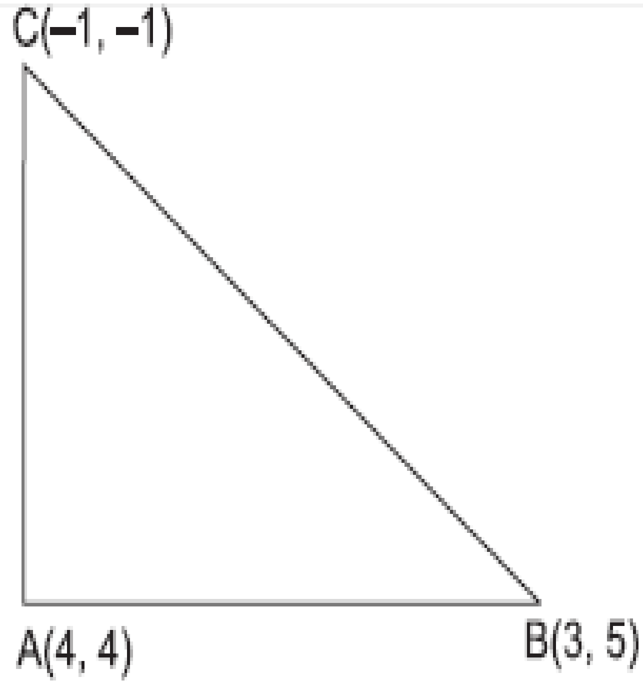
Hence, $x = 1$.

EXAMPLE . *Without using Pythagoras's theorem show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right-angled triangle.*

SOLUTION Let $A(4, 4)$, $B(3, 5)$ and $C(-1, -1)$ be the vertices of ΔABC.

Let m_1 and m_2 be the slopes of AB and AC respectively. Then,

$$m_1 = \text{slope of } AB = \frac{(5-4)}{(3-4)} = -1.$$



$m_2 = \text{slope of } AC = \dfrac{(-1-4)}{(-1-4)} = \dfrac{-5}{-5} = 1.$

$\therefore \quad m_1 m_2 = -1.$

So, $AB \perp AC$ and therefore, $\angle CAB = 90°$.

Hence, the given points are the vertices of a right triangle.

EXAMPLE *If the points (h, 0), (a, b) and (0, k) lie on a line, show that* $\frac{a}{h}+\frac{b}{k}=1$.

SOLUTION Let $A(h, 0)$, $B(a, b)$ and $C(0, k)$ be the given collinear points.

Since the given points A, B, C are collinear, we have

slope of AB = slope of BC

$$\therefore \quad \frac{b-0}{a-h}=\frac{k-b}{0-a} \Leftrightarrow \frac{b}{(a-h)}=\frac{(b-k)}{a}$$

$$\Leftrightarrow ab=(a-h)(b-k)$$

$$\Leftrightarrow ak + hb = hk$$

$$\Leftrightarrow \frac{a}{h} + \frac{b}{k} = 1 \qquad \text{[on dividing both sides by } hk\text{]}$$

Hence, $\frac{a}{h} + \frac{b}{k} = 1.$

EXAMPLE *If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.*

SOLUTION We know that the acute angle θ between two lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|. \quad \text{... (i)}$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$.

Putting these values in (i), we get

$$\left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = \tan\frac{\pi}{4} \Leftrightarrow \left|\frac{2m - 1}{2 + m}\right| = 1$$

$$\Leftrightarrow \left(\frac{2m-1}{2+m}\right) = \pm 1$$

$$\Leftrightarrow \frac{2m-1}{2+m} = 1 \quad \text{or} \quad \frac{2m-1}{2+m} = -1$$

$$\Leftrightarrow (2m-1) = (2+m) \quad \text{or} \quad (2m-1) = (-2-m)$$

$$\Leftrightarrow m = 3 \quad \text{or} \quad m = \frac{-1}{3}.$$

Hence, the slope of the other line is 3 or $\frac{-1}{3}$.

SUMMARY

(i) *Equation of x-axis is* $y = 0$.

(ii) *Equation of y-axis is* $x = 0$.

(iii) *Equation of a vertical line on RHS of the y-axis at a distance a from it is* $x = a$..

(iv) *Equation of a vertical line on LHS of the x-axis at a distance a from it is* $x = -a$..

(v) *Equation of a horizontal line lying above the x-axis at a distance b from it is* $y = b$.

(vi) *Equation of a horizontal line lying below the x-axis at a distance b from it is* $y = -b$.

Lecture- 23,24

General equation of line

General equation of a line:-

The equation $Ax+By+C=0$ always represent a line provided A and B are not simultaneously zero.

Reduction of General form to standard form.

Let the given equation of the line be

$$Ax + By + C = 0$$

Slope-Intercept form:

$$Ax + By + C = 0$$

$$\Rightarrow By = -Ax - C$$

$$\Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$$

$$\Rightarrow y = mx + c'$$

where $m = -\frac{A}{B}$, $c' = -\frac{C}{B}$

Intercepts form :

$$Ax + By + C = 0$$

$$Ax + By = -C$$

$$\frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\frac{x}{(-C/A)} + \frac{y}{(-C/B)} = 1$$

where x-intercept $= -\frac{C}{A}$, y-intercept $= -C/B$.

Normal form: $Ax + By + C = 0$

$$\Rightarrow Ax + By = -C$$

$$\Rightarrow \underbrace{\frac{-A}{\sqrt{A^2+B^2}}}_{\cos\alpha\ (\text{say})} x + \underbrace{\left(\frac{-B}{\sqrt{A^2+B^2}}\right)}_{\sin\alpha\ (\text{say})} y = \underbrace{+\frac{C}{\sqrt{A^2+B^2}}}_{p\ (\text{say})}$$

So it is converted into

$$x\cos\alpha + y\sin\alpha = p$$

Q-1 Reduce the following equations into normal form and find their perpendicular distance from the origin. [2018-19]

Sol.

(i) $x - \sqrt{3}y + 8 = 0$

(ii) $y - 2 = 0$

(i) $x - \sqrt{3}y = -8 \Rightarrow -x + \sqrt{3}y = 8$

$$-\frac{x}{\sqrt{1^2+(-\sqrt{3})^2}} + \frac{\sqrt{3}y}{\sqrt{(1)^2+(-\sqrt{3})^2}} = \frac{+8}{\sqrt{1^2+(-\sqrt{3})^2}}$$

$$\frac{-x}{\sqrt{4}} + \frac{\sqrt{3}y}{\sqrt{4}} = \frac{+8}{\sqrt{4}}$$

$$-\frac{x}{2} + \frac{\sqrt{3}}{2}y = +2$$

where, $\cos\alpha = \frac{-1}{2}$, $\sin\alpha = \frac{\sqrt{3}}{2}$, $P = 2$

Since $\cos\alpha < 0$ & $\sin\alpha > 0$, α lies in second quadrant.

Now $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\sqrt{3}/2}{-1/2} = -\sqrt{3} = \tan 60^\circ$

$$\tan\alpha = \tan(180 - 60^\circ) = \tan 120^\circ$$

$\Rightarrow$ $\alpha = 120^\circ$ & $p = 2$

Hence the equation in normal form is given by

$$\boxed{x\cos 120^\circ + y\sin 120^\circ = 2}$$

(ii) $y - 2 = 0$

$$0 \cdot x + y = 2$$

$$\frac{0 \cdot x}{\sqrt{1}} + \frac{y}{\sqrt{1}} = \frac{2}{\sqrt{1}}$$

where

$\cos\alpha = 0, \quad \sin\alpha = 1, \quad p = 2$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1}{0} = \infty = \tan\left(\frac{\pi}{2}\right)$$
$$= \tan 90^\circ$$

$$\alpha = 90^\circ, \; p = 2$$

Hence the required equation in normal form is given by

$$\boxed{x\cos 90^\circ + y\sin 90^\circ = 2}$$

Q-2 Reduce the equation $3x-2y+4=0$
in (i) slope-intercept form
(ii) intercept form.
(iii) normal form

(ii) Intercept form & (iii) Normal form, (i) slope intercept form

we have, $3x - 2y + 4 = 0$ (for slope-intercept form)

$\Rightarrow \quad 2y = 3x + 4$

$\Rightarrow \quad y = \frac{3}{2}x + \frac{4}{2}$

$\Rightarrow \quad y = \frac{3}{2}x + 2$

which is in slope intercept form

where $m = \frac{3}{2}$, y-intercept $= 2$

for intercept form-

$$3x - 2y = -4$$

$$\frac{3x}{(-4)} - \frac{2y}{(-4)} = 1$$

$$\frac{x}{(-4/3)} + \frac{y}{(2)} = 1$$

where $a = -4/3$, $b = 2$

which is required intercepts form.

for normal form, we have-

$$-3x+2y=4$$

$$-\frac{3}{\sqrt{9+4}}x+\frac{2y}{\sqrt{9+4}}=\frac{4}{\sqrt{3}}$$

$$\cos\alpha=\frac{-3}{\sqrt{13}},\quad \sin\alpha=\frac{2}{\sqrt{13}},\quad p=\frac{4}{\sqrt{3}}.$$

$$\tan\alpha=\left(\frac{-2}{3}\right)\Rightarrow \alpha=\tan^{-1}\left(-\frac{2}{3}\right)\ \&$$

$p=4/\sqrt{3}$ which is in form as normal form.

PRACTICE QUESTIONS

1. Reduce the equation $2x - 3y - 5 = 0$ to slope–intercept form, and find from it the slope and y-intercept.
2. Reduce the equation $5x + 7y - 35 = 0$ to slope–intercept form, and hence find the slope and the y-intercept of the line.
3. Reduce the equation $y + 5 = 0$ to slope–intercept form, and hence find the slope and the y-intercept of the line.
4. Reduce the equation $3x - 4y + 12 = 0$ to intercepts form. Hence, find the length of the portion of the line intercepted between the axes.
5. Reduce the equation $5x - 12y = 60$ to intercepts form. Hence, find the length of the portion of the line intercepted between the axes.
6. Find the inclination of the line:

   (i) $x + \sqrt{3}y + 6 = 0$ (ii) $3x + 3y + 8 = 0$ (iii) $\sqrt{3}x - y - 4 = 0$

ANSWERS

1. $y = \frac{2}{3}x - \frac{5}{3}, m = \frac{2}{3}$ and $c = \frac{-5}{3}$

2. $y = \frac{-5}{7}x + 5, m = \frac{-5}{7}$ and $c = 5$

3. $y = 0 \cdot x - 5, m = 0$ and $c = -5$

4. $\frac{x}{-4} + \frac{y}{3} = 1, 5$ units

5. $\frac{x}{12} + \frac{y}{-5} = 1, 13$ units

6. (i) 150° (ii) 135° (iii) 60°

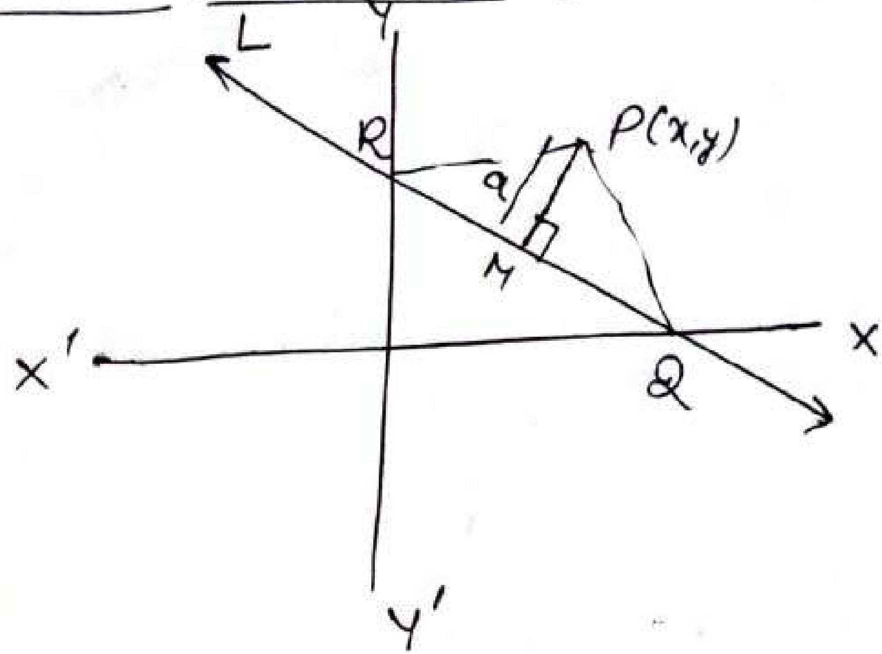
- ***<u>THANK YOU</u>***

Lecture- 25

Distance of a point from a line and distance between two parallel lines

Distance of a point from a line :-

The length of perpendicular from a given point $P(x_1, y_1)$ on a line $Ax+By+C=0$ is given by



On a line

$Ax+By+C=0$ is

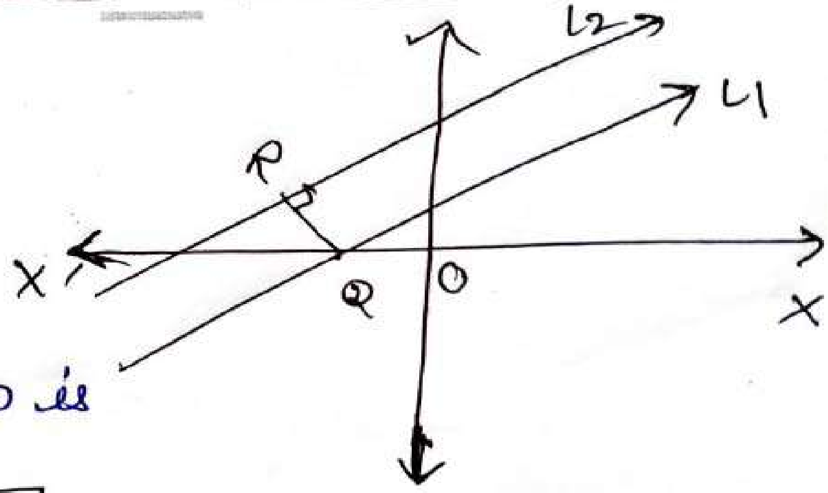
given by

$$d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

Distance between two parallel lines -

The distance between two parallel lines $Ax+By+C_1=0$ & $Ax+By+C_2=0$ is given by

$$d=\frac{|C_2-C_1|}{\sqrt{A^2+B^2}}$$



* Distance between two parallel lines $y = mx + c_1$ & $y = mx + c_2$ is given by

$$d = \frac{|c_2 - c_1|}{\sqrt{1+m^2}}$$

(1) Find the distance of the point (4,1) from the line $3x-4y+12=0$

Sol. Let the required distance be d then

$$d = \frac{|3\times 4 - 4\times 1 + 12|}{\sqrt{3^2+(-4)^2}} = \frac{20}{5} = 4 \text{ units.}$$

② If the lines are $3x-4y+9=0$ & $6x-8y-17=0$, find the distance between them.

Sol. $3x - 4y + 9 = 0 \Rightarrow y = \frac{3}{4}x + \frac{9}{4} \longrightarrow (1)$

$6x - 8y - 17 = 0 \Rightarrow y = \frac{3}{4}x - \frac{17}{8} \longrightarrow (2)$

Both lines are parallel (same slope) so

distance between given lines $= \frac{|c_2 - c_1|}{\sqrt{1+m^2}}$

where $m = \frac{3}{4}$,

$c_1 = 9/4$ & $c_2 = \frac{-17}{8}$

$= \frac{\left|\frac{-17}{8} - \frac{9}{4}\right|}{\sqrt{1+(3/4)^2}}$

$= \frac{\left|\frac{-35}{8}\right|}{\sqrt{1+\frac{9}{16}}} = \frac{\frac{35}{8}}{\frac{5}{4}} = \frac{7}{2}$ units

Ans.

EXAMPLE: *Find the length of perpendicular from the point (a, b) to the line $\frac{x}{a} + \frac{y}{b} = 1$.*

SOLUTION The given point is $P(a, b)$ and the given line is $bx + ay - ab = 0$.

Let d be the length of perpendicular from $P(a, b)$ to the line $bx + ay - ab = 0$.

Then, $d = \frac{|b \times a + a \times b - ab|}{\sqrt{b^2 + a^2}} = \frac{|ab|}{\sqrt{a^2 + b^2}}$ units.

EXAMPLE . *Find the length of perpendicular from the origin to the line $4x + 3y - 2 = 0$.*

SOLUTION The given point is $P(0, 0)$ and the given line is $4x + 3y - 2 = 0$.

Led d be the length of perpendicular from $P(0, 0)$ to the line $4x + 3y - 2 = 0$.

$$\text{Then, } d = \frac{|4 \times 0 + 3 \times 0 - 2|}{\sqrt{4^2 + 3^2}} = \frac{2}{5} \text{ unit.}$$

EXAMPLE *If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b then show that*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

SOLUTION The equation of the line making intercepts a and b on the axes is given by

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ i.e., } \frac{x}{a} + \frac{y}{b} - 1 = 0. \qquad \text{... (i)}$$

Since p is the length of perpendicular from $O(0, 0)$ to line (i), we have

$$p = \frac{\left|\frac{1}{a} \times 0 + \frac{1}{b} \times 0 - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

$$\Rightarrow \quad p^2 = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \frac{a^2b^2}{(b^2 + a^2)}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{(b^2 + a^2)}{a^2b^2} = \left(\frac{b^2}{a^2b^2} + \frac{a^2}{a^2b^2}\right) = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Hence, $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

EXAMPLE : *Find the distance between the parallel lines* $15x + 8y - 34 = 0$ *and* $15x + 8y + 31 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$15x + 8y - 34 = 0 \Rightarrow y = \frac{-15}{8}x + \frac{17}{4} \quad \text{... (i)}$$

$$15x + 8y + 31 = 0 \Rightarrow y = \frac{-15}{8}x - \frac{31}{8} \quad \text{... (ii)}$$

Clearly, the slopes of the given lines are equal and so they are parallel.

The given lines are of the form $y = mx + C_1$ and $y = mx + C_2$, where $m = \frac{-15}{8}$, $C_1 = \frac{17}{4}$ and $C_2 = \frac{-31}{8}$.

$\therefore$ distance between the given lines

$$= \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}, \text{ where } m = \frac{-15}{8}, C_1 = \frac{17}{4} \text{ and } C_2 = \frac{-31}{8}$$

$$= \frac{\left|\frac{-31}{8} - \frac{17}{4}\right|}{\sqrt{1 + \left(\frac{-15}{8}\right)^2}} = \frac{\left|-\frac{65}{8}\right|}{\sqrt{1 + \frac{225}{64}}} = \frac{\left(\frac{65}{8}\right)}{\sqrt{\frac{289}{64}}} = \left(\frac{65}{8} \times \frac{8}{17}\right) = \frac{65}{17} \text{ units.}$$

Hence, the distance between the given lines is $\frac{65}{17}$ units.

PRACTICE QUESTIONS

1. Find the distance of the point $(3, -5)$ from the line $3x - 4y = 27$.

2. Find the distance of the point $(-2, 3)$ from the line $12x = 5y + 13$.

3. Find the distance of the point $(-4, 3)$ from the line $4(x + 5) = 3(y - 6)$.

4. Find the distance of the point $(2, 3)$ from the line $y = 4$.

PRACTICE QUESTIONS

Prove that the line $12x-5y-3=0$ is mid-parallel to the lines $12x-5y+7=0$ and $12x-5y-13=0$.

ANSWERS

1. $\frac{2}{5}$ unit 2. 4 units 3. $\frac{13}{5}$ units 4. 1 unit

- ***THANK YOU***

Lecture- 26

Standard Equation of circle and its based questions

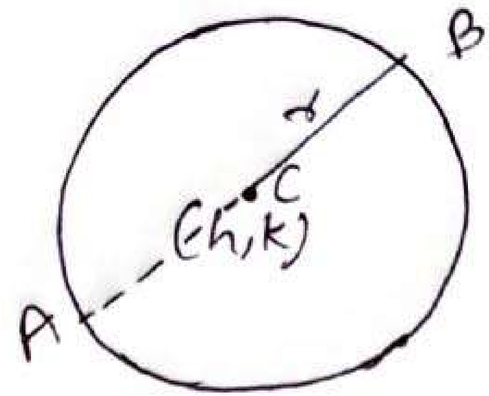
Standard equation of Circle :-

$C \equiv$ Centre (h,k)

r = radius

AB = diameter

= 2 x radius



The equation of a circle with centre (h,k) and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

* If centre = (0,0) orgin

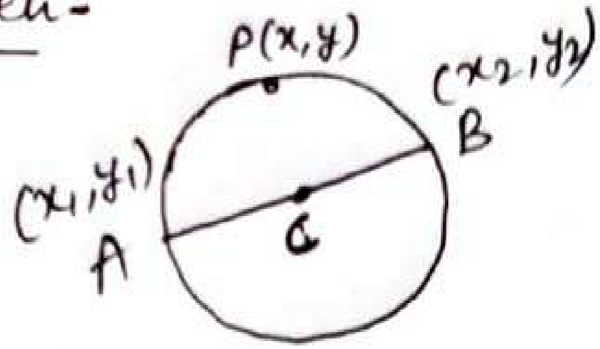
So $\boxed{x^2+y^2=r^2}$ is the required equation of circle.

* If centre = (0,0) origin

So $x^2+y^2=r^2$ is the required equation of circle.

Equation of a circle, the end points of whose diameter are given-

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$



which is required equation of circle

Q-1 Does the point $\left(\frac{5}{2}, \frac{7}{2}\right)$ lie inside, Outside or on the circle $(x^2+y^2=25)$?

[2011-12]

Sol. If centre $C(h,k)$ & given point (x_1, y_1) & radius $= r$ then distance between (x_1, y_1) & (h, k) = d_1

If $d_1 < r$ (point lies inside the circle)

$d_1 = r$ (point lies on the circle)

$d_1 > r$ (point lies outside the circle)

So we have, $(h, k) \equiv (0, 0)$, $\left(\frac{5}{2}, \frac{7}{2}\right) \equiv (x_1, y_1)$

$$d = \sqrt{\left(\frac{5}{2}-0\right)^2 + \left(\frac{7}{2}-0\right)^2}$$

$$d = \sqrt{\frac{25}{4} + \frac{49}{4}} = \sqrt{\frac{74}{4}} = \sqrt{74}/2 = 4.3011$$

radius = 5

radius = 5

$4.3011 < 5$

So the point $\left(\frac{5}{2}, \frac{7}{2}\right)$ lies inside the circle $x^2+y^2=25$ Ans.

Q-2 Find the equation of circle whose centre is (3, 2) and radius is 5. [2015-16]

Sol. The equation of circle having centre (3, 2) and radius 5 is

$$(x-3)^2 + (y-2)^2 = 5^2$$

$$(x-3)^2 + (y-2)^2 = 25$$

$$x^2 + 9 - 6x + y^2 + 4 - 4y = 25$$

$$x^2 + y^2 - 6x - 4y + 13 = 25$$

$$\boxed{x^2 + y^2 - 6x - 4y - 12 = 0}$$ Ans.

Q.3 Find the equation of circle whose centre is (2,3) and radius is 8. [2020-21]

Sol. The equation of a circle is

$$(x-2)^2 + (y-3)^2 = 8^2$$

$$x^2 + 4 - 4x + y^2 + 9 - 6y = 64$$

$$x^2 + y^2 - 4x - 6y + 13 = 64$$

$$\boxed{x^2 + y^2 - 4x - 6y - 51 = 0}$$ Ans.

Q.4 Find the equation of circle, the co-ordinates of whose diameter are (-1,2) and (4,-3). [2020-21]

Sol. The equation of circle is

$$(x+1)(x-4) + (y-2)(y+3) = 0$$

$$(x^2 - 4x + x - 4) + (y^2 + 3y - 2y - 6) = 0$$

$$\boxed{x^2 + y^2 - 3x + y - 10 = 0}$$ Ans.

EXAMPLE *Find the equation of a circle whose centre is (2, –1) and which passes through the point (3, 6).*

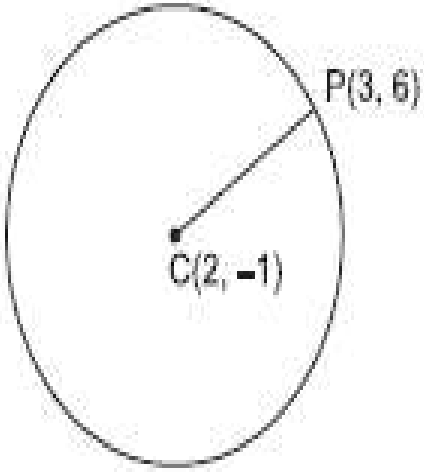
SOLUTION Let $C(2, -1)$ be the centre of the given circle and let it pass through the point $P(3, 6)$. Then, radius of the circle

$$= |CP| = \sqrt{(3-2)^2 + (6+1)^2} = \sqrt{50}.$$

$\therefore$ the required equation of the circle is

$$(x-2)^2 + (y+1)^2 = (\sqrt{50})^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 2y - 45 = 0.$$



EXAMPLE - *Find the equation of a circle of radius 5 units, whose centre lies on the x-axis and which passes through the point (2, 3).*

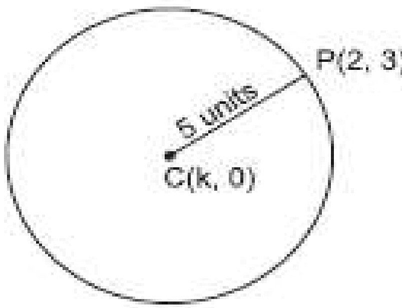
SOLUTION It is given that the centre of the circle lies on the x-axis.

So, let $C(k, 0)$ be the centre of the circle.

Also, it is given that it passes through the point $P(2, 3)$.

$\therefore$ radius of the circle $= |CP|$

$$= \sqrt{(k-2)^2 + (0-3)^2}$$
$$= \sqrt{k^2 - 4k + 13}$$



$\Rightarrow \sqrt{k^2 - 4k + 13} = 5$ [$\because$ radius = 5 units]

$\Rightarrow k^2 - 4k + 13 = 25 \Rightarrow k^2 - 4k - 12 = 0$

$\Rightarrow (k-6)(k+2) = 0 \Rightarrow k = 6$ or $k = -2$

$\therefore$ centre of the circle is $(6, 0)$ or $(-2, 0)$.

Hence, the required equation of the circle is

$$(x-6)^2 + (y-0)^2 = 5^2 \text{ or } (x+2)^2 + (y-0)^2 = 5^2$$

$\Rightarrow x^2 + y^2 - 12x + 11 = 0$ or $x^2 + y^2 + 4x - 21 = 0$.

Thus, there are two circles satisfying the given conditions.

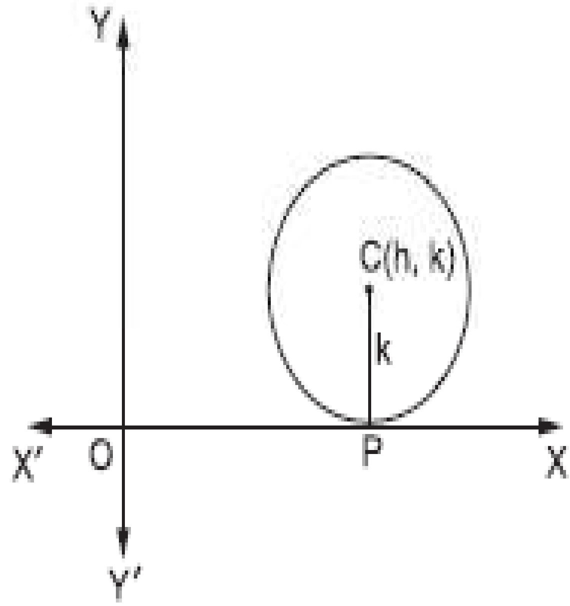
EXAMPLE *Find the equation of a circle with centre (h, k) and touching the x-axis.*

SOLUTION Clearly, the radius of the circle = k.

So, the required equation is

$$(x-h)^2 + (y-k)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 = 0.$$



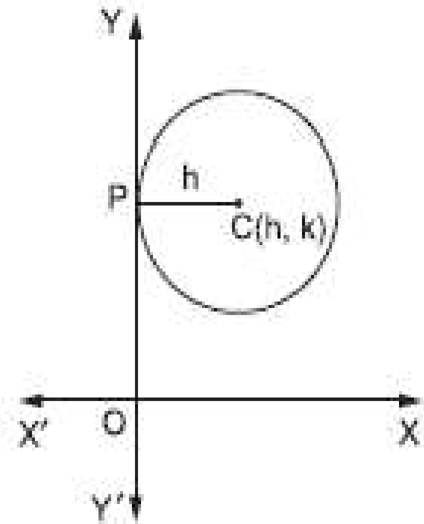
EXAMPLE 9 *Find the equation of a circle with centre (h, k) and touching the y-axis.*

SOLUTION Clearly, the radius of the circle = h.

So, the required equation is

$$(x-h)^2+(y-k)^2=h^2$$

$$\Rightarrow \quad x^2+y^2-2hx-2ky+k^2=0.$$



EXAMPLE _ *Find the equation of a circle with centre (h, k) and touching both the axes.*

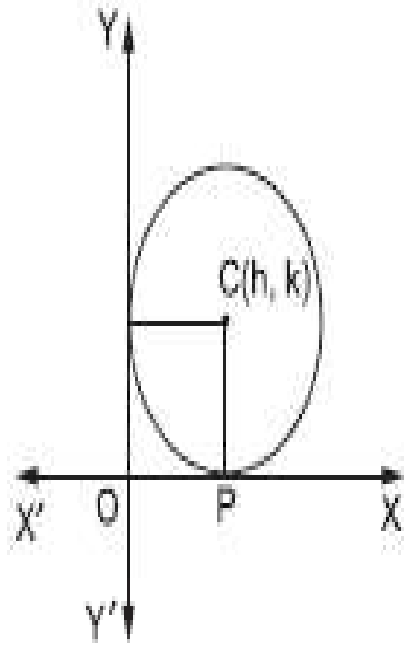
SOLUTION Clearly, radius $= h = k = c$ (say).

$\therefore$ the equation of the circle is

$$(x - c)^2 + (y - c)^2 = c^2$$

$$\Rightarrow \quad x^2 + y^2 - 2c(x + y) + c^2 = 0,$$

where $c = h = k$.



PRACTICE QUESTIONS

Find the equation of a circle with

1. centre (2, 4) and radius 5
2. centre (–3, –2) and radius 6
3. centre (a, a) and radius $\sqrt{2}$
4. centre $(a\cos\alpha, a\sin\alpha)$ and radius a
5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$
6. centre at the origin and radius 4
7. Find the centre and radius of each of the following circles:

   (i) $(x-3)^2 + (y-1)^2 = 9$

   (ii) $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{16}$

   (iii) $(x+5)^2 + (y-3)^2 = 20$

   (iv) $x^2 + (y-1)^2 = 2$

PRACTICE QUESTIONS

8. Find the equation of the circle whose centre is $(2, -5)$ and which passes through the point $(3, 2)$.

9. Find the equation of the circle of radius 5 cm, whose centre lies on the y-axis and which passes through the point $(3, 2)$.

10. Find the equation of the circle whose centre is $(2, -3)$ and which passes through the intersection of the lines $3x + 2y = 11$ and $2x + 3y = 4$.

ANSWERS

1. $x^2 + y^2 - 4x - 8y - 5 = 0$
2. $x^2 + y^2 + 6x + 4y - 23 = 0$
3. $x^2 + y^2 - 2ax - 2ay = 0$
4. $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$
5. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
6. $x^2 + y^2 - 16 = 0$
7. (i) Centre (3, 1), radius = 3 (ii) Centre $\left(\frac{1}{2}, \frac{-1}{3}\right)$, radius $= \frac{1}{4}$

   (iii) Centre (–5, 3), radius $= 2\sqrt{5}$ (iv) Centre (0, 1), radius $= \sqrt{2}$
8. $x^2 + y^2 - 4x + 10y - 21 = 0$
9. $(x^2 + y^2 - 12y + 11 = 0)$ or $(x^2 + y^2 + 4y - 21 = 0)$
10. $x^2 + y^2 - 4x + 6y + 3 = 0$

- ***THANK YOU***

Lecture- 27

Standard Equation of ellipse and its properties

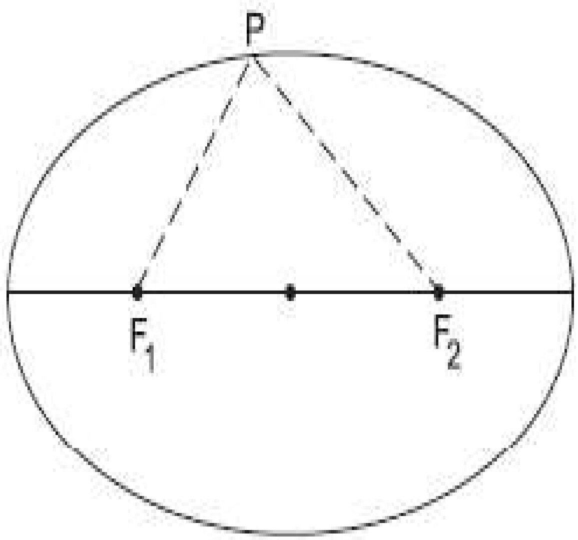
ELLIPSE *It is the path traced by a point which moves in a plane in such a way that the sum of its distance from two fixed points in the plane is a constant.*

The two fixed points are called the *foci* of the ellipse.

NOTE The plural of focus is foci.

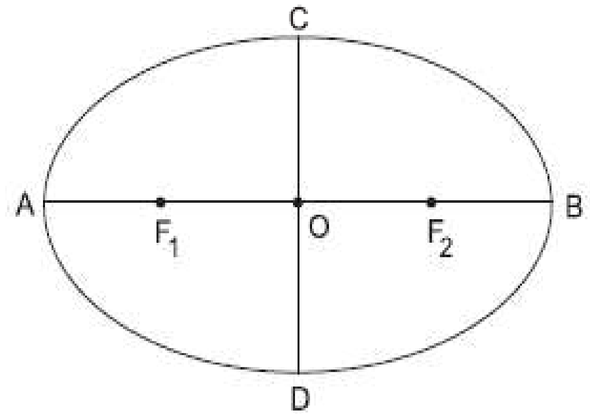
In the given figure, F_1 and F_2 are two fixed points and P is a point which moves in such a way that $PF_1 + PF_2 = constant$.

The path traced by the point P is called an ellipse, and the points F_1 and F_2 are called its *foci.*



SOME MORE TERMS RELATED TO AN ELLIPSE

(I) CENTRE OF THE ELLIPSE



The midpoint of the line segment joining the foci, is called the centre of the ellipse.

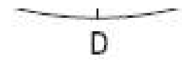
In the given figure, F_1 and F_2 are the foci of the ellipse and O is its *centre*, where $OF_1 = OF_2$.

(II) AXES OF THE ELLIPSE

MAJOR AXIS: *The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.*

In the given figure, AB is the major axis of the ellipse.

MINOR AXIS: *The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.*



(II) AXES OF THE ELLIPSE

MAJOR AXIS: *The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.*

In the given figure, AB is the major axis of the ellipse.

MINOR AXIS: *The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.*

In the given figure, CD is the minor axis of the ellipse.

(III) VERTICES OF AN ELLIPSE

The end points of the major axis of an ellipse are called its vertices.

In the given figure, A and B are the vertices of the ellipse.

AN IMPORTANT NOTE In an ellipse, we take:

Length of the major axis $= AB = 2a$.

Length of the minor axis $= CD = 2b$.

AN IMPORTANT NOTE In an ellipse, we take:

Length of the major axis $= AB = 2a$.

Length of the minor axis $= CD = 2b$.

Distance between the foci = $F_1F_2 = 2c$.

Length of the semi-major axis = a.

Length of the semi-minor axis = b.

(IV) ECCENTRICITY OF AN ELLIPSE

The ratio $\frac{c}{a}$ is always constant and it is denoted by e, called the *eccentricity* of the ellipse.

For an ellipse, we have $0 < e < 1$ $\left[\because\ c < a \Leftrightarrow e = \frac{c}{a} < 1\right]$.

STANDARD EQUATION OF AN ELLIPSE

THEOREM ... t the standard equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b are the lengths of the semi-major axis and the semi-minor axis respectively and $a > b$.

HORIZONTAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the larger denominator then its major axis lies along the x-axis.

Such an ellipse is called a *horizontal ellipse.*

Thus, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is an horizontal ellipse, if $a^2 > b^2$.

EXAMPLE $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ is an horizontal ellipse.

LATUS RECTUM OF A HORIZONTAL ELLIPSE

The latus rectum of an ellipse is a line segment perpendicular to the major axis, passing through any of the foci with end points lying on the ellipse.

a

the length of the latus rectum is $\frac{2b^2}{a}$.

In horizontal ellipse,	In vertical ellipse,
(1) Centre $\equiv (0,0)$	(1) Centre $\equiv (0,0)$
(2) Vertices: $A(-a,0)$ & $B(a,0)$	(2) Vertices: $A(0,-a)$, $B(0,a)$
(3) Foci: $F_1(-c,0)$ & $F_2(c,0)$	(3) Foci: $F_1(0,-c)$ & $F_2(0,c)$
(4) eccentricity $(e) = \frac{c}{a}$ where $c = \sqrt{a^2-b^2}$	(4) $e = \frac{c}{a}$ where $c = \sqrt{a^2-b^2}$

(5) length of major Axis (AB) – 2a

(6) length of minor axis (CD) – 2b

(7) Equation of major axis – $y = 0$

(5) length of major axis (AB) – 2a

(6) length of minor axis (CD) – 2b

(7) Equation of major axis – $x = 0$

(8) Equation of minor axis - $x = 0$

(9) Length of latus rectum - $\frac{2b^2}{a}$

(8) Equation of minor axis - $y = 0$

(9) Length of latus rectum - $\frac{2b^2}{a}$

(i) Tranverse Axis -
X'OX

(ii) Conjugate Axis -
YOY'

(iii) Foci ⇒ $(-c,0)$ & $(c,0)$

(iv) Vertices ⇒ $(-a,0)$ & $(a,0)$

(v) Length of tranverse axis $2a$ & equation is $y=0$.

(i) Transverse Axis -
YOY'

(ii) Conjugate axis -
X'OX

(iii) Foci ⇒ $(0,-c)$ & $(0,c)$

(iv) Vertices ⇒ $(0,-a)$ & $(0,a)$

(v) Length of transverse axis $2a$ & equation is $x=0$.

(vi) length of conjugate axis $2b$ & equation is $x=0$.	(vi) length of conjugate axis $2b$ & equation is $x=0$.
(vii) length of latus rectum =	(vii) length of latus rectum $= 2b^2/a$

(1) Find eccentricity, co-ordinates of foci and length of latus rectum for the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$ [2015-16]

$a^2=36, \quad b^2=16$

$a^2 > b^2$ (Horizontal ellipse)

Sol.

$$c^2 = a^2 - b^2$$

$$c^2 = 36 - 16 = 20$$

$$\text{eccentricity}(e) = \frac{c}{a} = \frac{\sqrt{20}}{6} \quad \text{Ans}$$

$$= \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3} \quad \text{Ans}$$

$$\text{length of latus rectum} = \frac{2b^2}{a} = \frac{2 \times 16}{6_{3}}$$

$$= \frac{16}{3} \quad \text{Ans.}$$

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(2) Find the lengths of the major and minor axes; co-ordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse :

$$4x^2+y^2=100$$

Sol. The given equation is

$$4x^2+y^2=100$$

$$\frac{x^2}{25}+\frac{y^2}{100}=1 \qquad a^2>b^2$$

(Vertical ellipse)

$b^2 = 25, \; a^2 = 100$

$c = \sqrt{a^2 - b^2} = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$a = 10, \; b = 5$

(i) length of major axis $= 2a = 20$ units.

length of minor axis $= 2b = 2 \times 5 = 10$ units.

(ii) Co-ordinates of the vertices –

$A(0, -a)$ & $B(0, a)$ that is

$A(0, -10)$ & $B(0, 10)$.

(iii) Co-ordinates of foci are

$F_1(0, -c)$ & $F_2(0, c)$

$F_1(0, -5\sqrt{3})$ & $F_2(0, 5\sqrt{3})$

(iv) Eccentricity $(e) = \frac{c}{a} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$

(v) Length of the latus rectum $= \frac{2b^2}{a}$

$= \frac{2 \times 25}{10} = 5$ units.

Ans.

EXAMPLE *Find the lengths of the major and minor axes; coordinates of the vertices and the foci, the eccentricity and length of the latus rectum of the ellipse:*

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

SOLUTION Given equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse.*

Now, $(a^2 = 16 \text{ and } b^2 = 9 \Rightarrow (a = 4 \text{ and } b = 3)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}.$

Thus, $a = 4$, $b = 3$ and $c = \sqrt{7}$.

(i) Length of the major axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the minor axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-\sqrt{7}, 0)$ and $F_2(\sqrt{7}, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{7}}{4}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 9)}{3}$ units $= \frac{9}{2}$ units.

EXAMPLE *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$4x^2 + 9y^2 = 144.$$

SOLUTION The given equation may be written as

$$\frac{x^2}{36} + \frac{y^2}{16} = 1.$$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse*.

Now, $(a^2 = 36$ and $b^2 = 16) \Rightarrow (a = 6$ and $b = 4)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}$.

Thus, $a = 6$, $b = 4$ and $c = 2\sqrt{5}$.

(i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 4)$ units $= 8$ units.

(ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-2\sqrt{5}, 0)$ and $F_2(2\sqrt{5}, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$.

a 10

EXAMPLE *Find the equation of an ellipse whose vertices are at* $(\pm 5, 0)$ *and foci at* $(\pm 4, 0)$.

SOLUTION Since the vertices of the given ellipse are on the x-axis, so it is a *horizontal ellipse.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2>b^2$.

Its vertices are $(\pm a, 0)$ and therefore, $a=5$.

Its foci are $(\pm c, 0)$ and therefore, $c=4$.

$\therefore \quad c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (25 - 16) = 9.$

Thus, $a^2 = 5^2 = 25$ and $b^2 = 9.$

Hence, the required equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$

EXAMPLE 6 *Find the equation of an ellipse whose foci are* $(\pm 4, 0)$ *and the eccentricity is* $\frac{1}{3}$.

SOLUTION Since the foci of the given ellipse are on the x-axis, so it is a *horizontal ellipse*.

Let the required equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \text{where } a^2>b^2.$$

Let its foci be $(\pm c, 0)$. Then, $c=4$.

Also, $e=\frac{c}{a} \Leftrightarrow a=\frac{c}{e}=\frac{4}{(1/3)}=12.$

Now, $c^2=(a^2-b^2) \Rightarrow b^2=(a^2-c^2)=(144-16)=128.$

$\therefore \quad a^2=(12)^2=144$ and $b^2=128.$

Hence, the required equation is $\frac{x^2}{144}+\frac{y^2}{128}=1.$

EXAMPLE 7 *Find the equation of an ellipse whose major axis lies on the x-axis and which passes through the points (4, 3) and (6, 2).*

which passes through the points (4, 3) and (6, 2).

SOLUTION Since the major axis of the ellipse lies on the x-axis, so it is a *horizontal ellipse*.

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a^2 > b^2). \quad \dots \text{(i)}$$

Since (4, 3) lies on (i), we have $\frac{16}{a^2} + \frac{9}{b^2} = 1.$ ... (ii)

Also, since (6, 2) lies on (i), we have $\frac{36}{a^2} + \frac{4}{b^2} = 1.$ ... (iii)

Putting $\frac{1}{a^2} = u$ and $\frac{1}{b^2} = v$ in (ii) and (iii), we get

$$16u + 9v = 1 \quad \dots \text{(iv)}$$

$$36u + 4v = 1 \quad \dots \text{(v)}$$

On multiplying (iv) by 9 and (v) by 4, and subtracting, we get

$$65v = 5 \Leftrightarrow v = \frac{1}{13} \Leftrightarrow \frac{1}{b^2} = \frac{1}{13} \Leftrightarrow b^2 = 13.$$

Also, since (6, 2) lies on (i), we have $\frac{36}{a^2} + \frac{4}{b^2} = 1.$ ... (iii)

Putting $\frac{1}{a^2} = u$ and $\frac{1}{b^2} = v$ in (ii) and (iii), we get

$$16u + 9v = 1 \quad \text{... (iv)}$$

$$36u + 4v = 1 \quad \text{... (v)}$$

On multiplying (iv) by 9 and (v) by 4, and subtracting, we get

$$65v = 5 \Leftrightarrow v = \frac{1}{13} \Leftrightarrow \frac{1}{b^2} = \frac{1}{13} \Leftrightarrow b^2 = 13.$$

Putting $v = \frac{1}{13}$ in (iv), we get

$$16u = \left(1 - \frac{9}{13}\right) \Leftrightarrow 16u = \frac{4}{13} \Leftrightarrow u = \left(\frac{4}{13} \times \frac{1}{16}\right) = \frac{1}{52}$$

$$\Leftrightarrow \frac{1}{a^2} = \frac{1}{52} \Leftrightarrow a^2 = 52.$$

Thus, $a^2 = 52$ and $b^2 = 13$.

Hence, the required equation is $\dfrac{x^2}{52} + \dfrac{y^2}{13} = 1$.

PRACTICE QUESTIONS

Find the (i) lengths of major and minor axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity, and (v) length of the latus rectum of each of the following ellipses.

1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
2. $\frac{x^2}{49} + \frac{y^2}{36} = 1$
3. $16x^2 + 25y^2 = 400$
4. $x^2 + 4y^2 = 100$
5. $9x^2 + 16y^2 = 144$
6. $4x^2 + 9y^2 = 1$
7. $\frac{x^2}{4} + \frac{y^2}{25} = 1$
8. $\frac{x^2}{9} + \frac{y^2}{16} = 1$
9. $3x^2 + 2y^2 = 18$
10. $9x^2 + y^2 = 36$
11. $16x^2 + y^2 = 16$
12. $25x^2 + 4y^2 = 100$

ANSWERS

1. (i) 10 units, 6 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-4, 0)$ and $F_2(4, 0)$
   (iv) $e = \frac{4}{5}$ (v) 3.6 units
2. (i) 14 units, 12 units (ii) $A(-7, 0)$ and $B(7, 0)$
   (iii) $F_1(-\sqrt{13}, 0)$ and $F_2(\sqrt{13}, 0)$ (iv) $e = \frac{\sqrt{13}}{7}$ (v) $\frac{72}{7}$ units
3. (i) 10 units, 8 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-3, 0)$ and $F_2(3, 0)$
   (iv) $e = \frac{3}{5}$ (v) $\frac{32}{5}$ units
4. (i) 20 units, 10 units (ii) $A(-10, 0)$ and $B(10, 0)$
   (iii) $F_1(-5\sqrt{3}, 0)$ and $F_2(5\sqrt{3}, 0)$ (iv) $e = \frac{\sqrt{3}}{2}$ (v) 5 units
5. (i) 8 units, 6 units (ii) $A(-4, 0)$ and $B(4, 0)$ (iii) $F_1(-\sqrt{7}, 0)$, $F_2(\sqrt{7}, 0)$
   (iv) $e = \frac{\sqrt{7}}{4}$ (v) $\frac{9}{2}$ units
6. (i) 1 unit, $\frac{2}{3}$ unit (ii) $A\left(-\frac{1}{2}, 0\right)$ and $B\left(\frac{1}{2}, 0\right)$
   (iii) $F_1\left(\frac{-\sqrt{5}}{6}, 0\right)$ and $F_2\left(\frac{\sqrt{5}}{6}, 0\right)$ (iv) $e = \frac{\sqrt{5}}{3}$ (v) $\frac{4}{9}$ unit

ANSWERS

7. (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$

   (iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $\frac{8}{5}$ units

8. (i) 8 units, 6 units (ii) $A(0, -4)$ and $B(0, 4)$

   (iii) $F_1(0, -\sqrt{7})$ and $F_2(0, \sqrt{7})$ (iv) $e = \frac{\sqrt{7}}{4}$ (v) $4\frac{1}{2}$ units

9. (i) 6 units, $2\sqrt{6}$ units (ii) $A(0, -3)$ and $B(0, 3)$

   (iii) $F_1(0, -\sqrt{3})$ and $F_2(0, \sqrt{3})$ (iv) $e = \frac{1}{\sqrt{3}}$ (v) 4 units

ANSWERS

10. (i) 12 units, 4 units

(ii) $A(0, -6)$ and $B(0, 6)$

(iii) $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$

(iv) $e = \dfrac{2\sqrt{2}}{3}$

(v) $1\dfrac{1}{3}$ units

11. (i) 8 units, 2 units

(ii) $A(0, -4)$ and $B(0, 4)$

(iii) $F_1(0, -\sqrt{15})$ and $F_2(0, \sqrt{15})$

(iv) $e = \dfrac{\sqrt{15}}{4}$

(v) $\dfrac{1}{2}$ unit

12. (i) 10 units, 4 units

(ii) $A(0, -5)$ and $B(0, 5)$

(iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$

(iv) $e = \dfrac{\sqrt{21}}{5}$

(v) $1\dfrac{3}{5}$ units

-
- THANK YOU

Lecture- 28

Standard Equation of hyperbola and its properties

HYPERBOLA *It is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.*

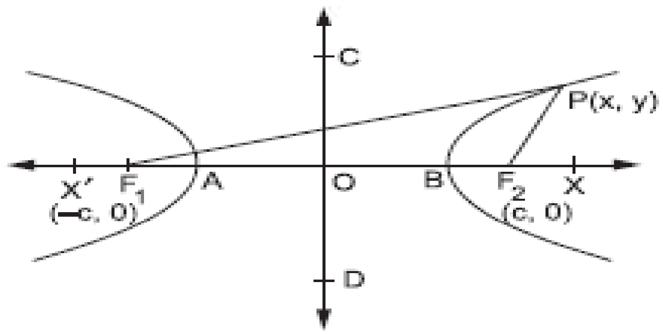
The two fixed points are called the *foci* of the hyperbola.

SOME MORE TERMS RELATED TO A HYPERBOLA

(I) CENTRE OF THE HYPERBOLA

The midpoint of the line segment joining the foci is called the centre of the hyperbola.

In the given figure, F_1 and F_2 are the foci of the hyperbola and O is its centre, where $OF_1 = OF_2$.



(II) AXES OF THE HYPERBOLA

TRANSVERSE AXIS: *The line through the foci of the hyperbola is called its transverse axis.*

In the given figure, $X'OX$ is the transverse axis of the hyperbola.

CONJUGATE AXIS: *The line through the centre and perpendicular to the transverse axis of the hyperbola is called its conjugate axis.*

Thus, COD is the conjugate axis of the hyperbola in the given figure.

(III) VERTICES OF THE HYPERBOLA

The points at which the hyperbola intersects the transverse axis are called its vertices.

In the given figure, A and B are the vertices of the hyperbola.

(IV) LENGTH OF TRANSVERSE AXIS

The distance between the two vertices of a hyperbola is called the length of its transverse axis.

In the given hyperbola, length of the transverse axis = AB.

AN IMPORTANT NOTE In a hyperbola, we take:

Length of the transverse axis $= AB = 2a$.

Distance between the foci $= F_1F_2 = 2c$, where $c > a$.

Length of the conjugate axis $= 2b$, where $b^2 = (c^2 - a^2)$.

(V) ECCENTRICITY OF A HYPERBOLA

The ratio $\frac{c}{a}$ *is always constant, called the eccentricity of the hyperbola and it is denoted by e.*

$$\text{Here, } c > a \Leftrightarrow \frac{c}{a} > 1 \Leftrightarrow e > 1.$$

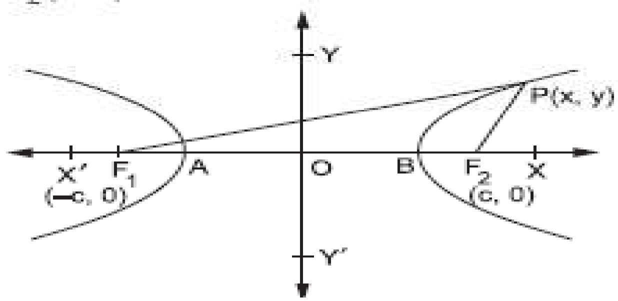
STANDARD EQUATION OF A HYPERBOLA

the standard equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

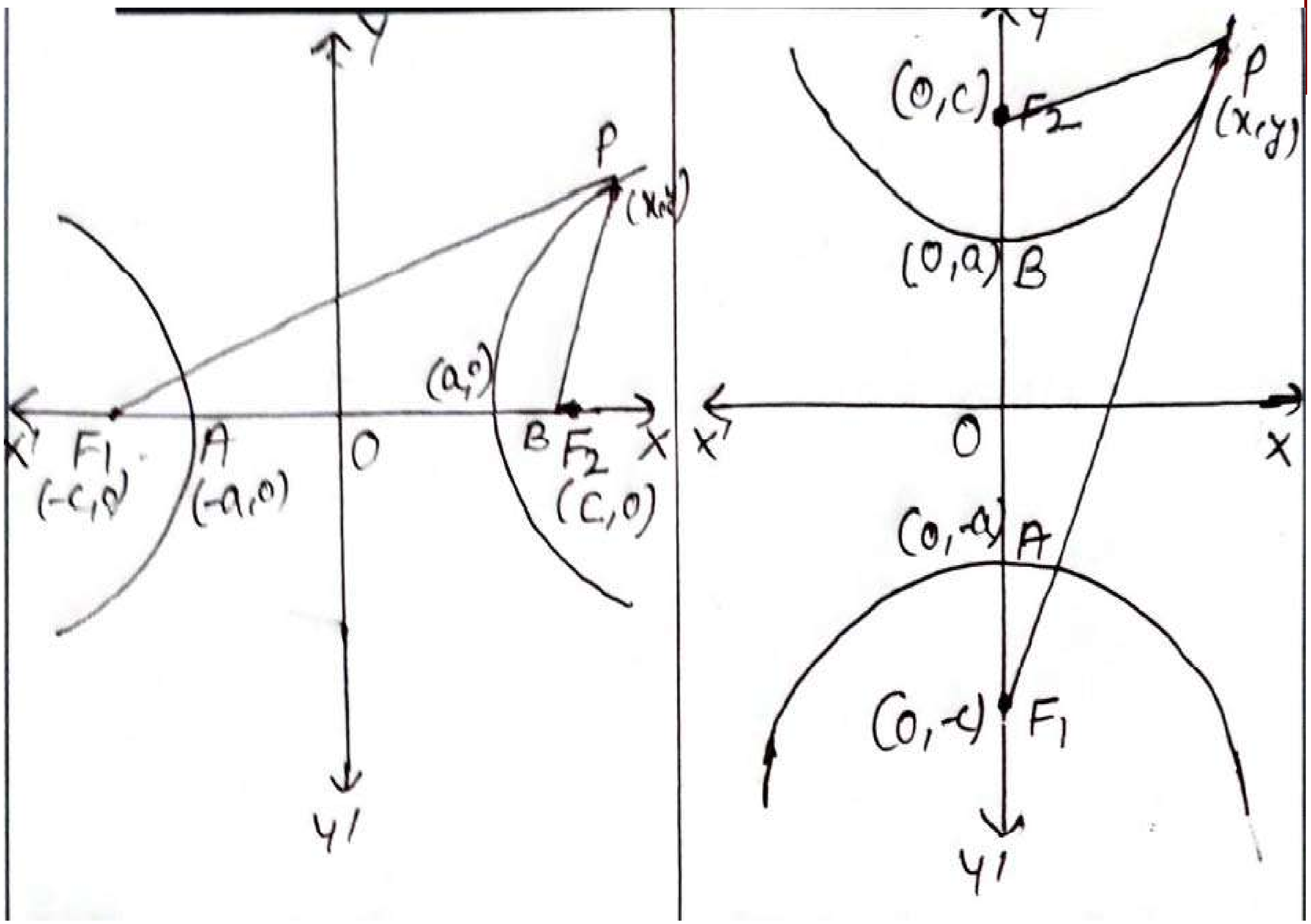
PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let us consider a hyperbola with centre at $O(0, 0)$ and its foci at $F_1(-c, 0)$ and $F_2(c, 0)$.



LATUS RECTUM OF A HYPERBOLA

The latus rectum of a hyperbola is a line segment perpendicular to the transverse axis, through any of the foci with its end points lying on the hyperbola.


Y
P
(x,y)
(a,0)
X'
F1
(-c,0)
A
(-a,0)
O
B
F2
(c,0)
X
Y'
Y
(0,c)
F2
P
(x,y)
(0,a)
B
X'
O
X
(0,-a)
A
(0,-c)
F1
Y'

(i) Tranverse Axis -
X'OX

(ii) Conjugate Axis -
YOY'

(iii) Foci ⇒ $(-c,0)$ & $(c,0)$

(iv) Vertices ⇒ $(-a,0)$ & $(a,0)$

(v) Length of tranverse axis $2a$ & equation is $y=0$.

(i) Transverse Axis -
YOY'

(ii) Conjugate axis -
X'OX

(iii) Foci ⇒ $(0,-c)$ & $(0,c)$

(iv) Vertices ⇒ $(0,-a)$ & $(0,a)$

(v) Length of transverse axis $2a$ & equation is $x=0$.

(vi) length of conjugate axis 2b & equation is $x=0$.	(vi) length of conjugate axis 2b & equation is $x=0$.
(vii) length of latus rectum =	(vii) length of latus rectum $= 2b^2/a$

Q.1 Find the equation of the hyperbola whose foci are $(0, \pm 12)$ and the length of the latus rectum is 36. [2011-12]

Sol. Foci $\equiv (0, \pm 12)$

$\Rightarrow$ Vertical Hyperbola

$c = 12$, $c^2 = a^2 + b^2$

$144 = a^2 + b^2 \longrightarrow (1)$

Given as length of latus rectum $= 36$

$$\frac{2b^2}{a} = 36$$

$$b^2 = 18a \longrightarrow (2)$$

By (1) & (2)

$$a^2 + 18a - 144 = 0$$

$$(a+24)(a-6) = 0$$

$$a = 6$$

$a \neq -ve$

By (2) $b^2 = 18 \times 6$

$$b^2 = 108$$

So the required equation of hyperbola is

$$\frac{y^2}{36} - \frac{x^2}{108} = 1$$

Since $a^2 = 36$, $b^2 = 108$.

Q.2 Find the equation of hyperbola having directrix $x+2y=1$, focus $(2,1)$ & eccentricity 2. [2013-14]

Sol.

$$PS = ePM$$

Let $P \equiv (x, y)$

$$(PS)^2 = e^2 (PM)^2$$

$$(x-2)^2 + (y-1)^2 = 4\left[\frac{x+2y-1}{\sqrt{1+4}}\right]^2$$

$$x^2 + 4 - 4x + y^2 + 1 - 2y = \frac{4}{(\sqrt{5})^2}(x+2y-1)^2$$

$$5x^2 + 20 - 20x + 5y^2 + 5 - 10y = 4(x^2 + 4y^2 + 1 + 4xy - 4y - 2x)$$

$$\Rightarrow \boxed{x^2 - 11y^2 - 16xy - 12x + 6y + 21 = 0}$$

which is required equation of hyperbola.

EXAMPLE *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$\frac{x^2}{36} - \frac{y^2}{64} = 1.$$

SOLUTION The equation of the given hyperbola is $\frac{x^2}{36} - \frac{y^2}{64} = 1.$

Comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 36$ and $b^2 = 64$.

$\therefore$ $a = 6$, $b = 8$ and $c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$

(i) Length of the transverse axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the conjugate axis $= 2b = (2 \times 8)$ units $= 16$ units.

(ii) The coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iii) The coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-10, 0)$ and $F_2(10, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 64}{6}\right)$ units $= \frac{64}{3}$ units.

EXAMPLE *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$9x^2 - 16y^2 = 144.$$

SOLUTION $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1.$

Thus, the equation of the given hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

Comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 16$ and $b^2 = 9$.

$\therefore$ $a = 4$, $b = 3$ and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$

(i) Length of the transverse axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the conjugate axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) The coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iii) The coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-5, 0)$ and $F_2(5, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{5}{4}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 9}{4}\right)$ units $= \frac{9}{2}$ units.

EXAMPLE *Find the equation of the hyperbola whose foci are $(\pm 5, 0)$ and the transverse axis is of length 8.*

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c, 0)$, it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Length of its transverse axis = $2a$.

$\therefore \quad 2a = 8 \Leftrightarrow a = 4 \Leftrightarrow a^2 = 16$.

Let its foci be $(\pm c, 0)$.

Then, $c = 5 \qquad [\because$ foci are $(\pm 5, 0)]$.

$\therefore \quad b^2 = (c^2 - a^2) = (5^2 - 4^2) = (25 - 16) = 9$.

Thus, $a^2 = 16$ and $b^2 = 9$.

Hence, the required equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

EXAMPLE _ *Find the equation of the hyperbola whose foci are at* $(0, \pm 6)$ *and the length of whose conjugate axis is* $2\sqrt{11}$.

SOLUTION Since the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Let its foci be $(0, \pm c)$.

But, the foci are at $(0, \pm 6)$.

$\therefore \quad c = 6.$

Length of the conjugate axis $= 2\sqrt{11}$

$$2b = 2\sqrt{11} \iff b = \sqrt{11} \iff b^2 = 11.$$

Also, $c^2 = (a^2 + b^2) \iff a^2 = (c^2 - b^2) = (36 - 11) = 25.$

Thus, $a^2 = 25$ and $b^2 = 11$.

Hence, the required equation is $\dfrac{y^2}{25} - \dfrac{x^2}{11} = 1.$

PRACTICE QUESTIONS

Find the (i) lengths of the axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity and (iv) length of the latus rectum of each of the following the hyperbola:

1. $\frac{x^2}{9} - \frac{y^2}{16} = 1$
2. $\frac{x^2}{25} - \frac{y^2}{4} = 1$
3. $x^2 - y^2 = 1$
4. $3x^2 - 2y^2 = 6$

PRACTICE QUESTIONS

5. $25x^2 - 9y^2 = 225$

6. $24x^2 - 25y^2 = 600$

7. $\frac{y^2}{16} - \frac{x^2}{49} = 1$

8. $\frac{y^2}{9} - \frac{x^2}{27} = 1$

9. $3y^2 - x^2 = 108$

10. $5y^2 - 9x^2 = 36$

11. Find the equation of the hyperbola with vertices at $(\pm 6, 0)$ and foci at $(\pm 8, 0)$.

12. Find the equation of the hyperbola with vertices at $(0, \pm 5)$ and foci at $(0, \pm 8)$.

ANSWERS

1. (i) 6 units, 8 units (ii) $A(-3, 0)$, $B(3, 0)$ (iii) $F_1(-5, 0)$, $F_2(5, 0)$

(iv) $e = \frac{5}{3}$ (v) $10\frac{2}{3}$ units

ANSWERS

2. (i) 10 units, 4 units (ii) $A(-5, 0)$, $B(5, 0)$ (iii) $F_1(-\sqrt{29}, 0)$, $F_2(\sqrt{29}, 0)$
   (iv) $e = \frac{\sqrt{29}}{5}$ (v) $1\frac{3}{5}$ units
3. (i) 2 units, 2 units (ii) $A(-1, 0)$, $B(1, 0)$ (iii) $F_1(-\sqrt{2}, 0)$, $F_2(\sqrt{2}, 0)$
   (iv) $e = \sqrt{2}$ (v) 2 units
4. (i) $2\sqrt{2}$ units, $2\sqrt{3}$ units (ii) $A(-\sqrt{2}, 0)$, $B(\sqrt{2}, 0)$ (iii) $F_1(-\sqrt{5}, 0)$, $F_2(\sqrt{5}, 0)$
   (iv) $e = \sqrt{\frac{5}{2}}$ (v) $3\sqrt{2}$ units
5. (i) 6 units, 10 units (ii) $A(-3, 0)$, $B(3, 0)$ (iii) $F_1(-\sqrt{34}, 0)$, $F_2(\sqrt{34}, 0)$
   (iv) $e = \frac{\sqrt{34}}{3}$ (v) $16\frac{2}{3}$ units
6. (i) 10 units, $4\sqrt{6}$ units (ii) $A(-5, 0)$, $B(5, 0)$ (iii) $F_1(-7, 0)$, $F_2(7, 0)$
   (iv) $e = 1\frac{2}{5}$ (v) $9\frac{3}{5}$ units
7. (i) 8 units, 14 units (ii) $A(0, -4)$, $B(0, 4)$ (iii) $F_1(0, -\sqrt{65})$, $F_2(0, \sqrt{65})$
   (iv) $e = \frac{\sqrt{65}}{4}$ (v) $24\frac{1}{2}$ units
8. (i) 6 units, $6\sqrt{3}$ units (ii) $A(0, -3)$, $B(0, 3)$ (iii) $F_1(0, -6)$, $F_2(0, 6)$
   (iv) $e = 2$ (v) 18 units

ANSWERS

9. (i) 12 units, $12\sqrt{3}$ units (ii) $A(0, -6)$, $B(0, 6)$ (iii) $F_1(0, -12)$, $F_2(0, 12)$

(iv) $e = 2$ (v) 36 units

10. (i) $\frac{12}{\sqrt{5}}$ units, 4 units (ii) $A\left(0, -\frac{6}{\sqrt{5}}\right)$, $B\left(0, \frac{6}{\sqrt{5}}\right)$

(iii) $F_1\left(0, -2\sqrt{\frac{14}{5}}\right)$, $F_2\left(0, 2\sqrt{\frac{14}{5}}\right)$ (iv) $e = \frac{\sqrt{14}}{3}$ (v) $\frac{4\sqrt{5}}{3}$ units

11. $\frac{x^2}{36} - \frac{y^2}{28} = 1$

12. $\frac{y^2}{25} - \frac{x^2}{39} = 1$

- ***<u>THANK YOU</u>***