UNIT-5

Lecture- 39

Rolls theorem & its Geometric Interpretations

Rolle's Theorem

Let f be a real-valued function, defined in the closed interval $[a, b]$ *such that*

(i) f *is continuous on* $[a, b]$; (ii) f *is differentiable on* $]a, b[$;

(iii) $f(a) = f(b)$.

Then, there exists a real number c in the open interval $]a, b[$ *such that*

$f'(c) = 0$

GEOMETRICAL SIGNIFICANCE OF ROLLE'S THEOREM Let f be a real function defined on $[a, b]$ and let Rolle's theorem be applicable on it. Then, f being continuous on $[a, b]$, it follows that we can draw a graph of $f(x)$ between the values $x = a$ and $x = b$.

Also, $f(x)$ being differentiable in $]a, b[$, it follows that the graph of $f(x)$ has a tangent at each point of $]a, b[$.

Now, the existence of a real number $c \in]a, b[$ such that $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has a slope 0, i.e., it is parallel to the x-axis.

EXAMPLE 1 *Verify Rolle's theorem for the function* $f(x) = x^3 - 6x^2 + 11x - 6$ *in the interval* $[1, 3]$.

SOLUTION Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 3]$.

(ii) $f'(x) = 3x^2 - 12x + 11$, which clearly exists for all values of $x \in [1, 3]$.

So, $f(x)$ is differentiable on the open interval $]1, 3[$.

(iii) $f(1) = (1^3 - 6 \times 1^2 + 11 \times 1 - 6) = 0$

and $f(3) = (3^3 - 6 \times 3^2 + 11 \times 3 - 6) = 0$.

$\therefore \quad f(1) = f(3)$.

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c \in]1, 3[$ such that $f'(c) = 0$.

Now, $\quad f'(c) = 0 \Rightarrow \; 3c^2 - 12c + 11 = 0$

$$\Rightarrow \; c = \frac{12 \pm \sqrt{144 - 132}}{6} \;\Rightarrow\; c = \left(2 \pm \frac{1}{\sqrt{3}}\right).$$

Clearly, both the values of c lie in the interval $]1, 3[$.

Hence, Rolle's theorem is verified.

EXAMPLE 2 *Verify Rolle's theorem for the function* $f(x) = x(x-1)^2$ *in the interval* $[0, 1]$.

SOLUTION We have $f(x) = x^3 - 2x^2 + x$.

We observe here that

(i) $f(x)$ being a polynomial function, is continuous on $[0,\ 1]$.

(ii) $f'(x) = (3x^2 - 4x + 1)$, which clearly exists for all values of $x \in [0,\ 1]$.

So, $f(x)$ is differentiable on the interval $]0, 1[$.

(iii) $f(0) = 0$ and $f(1) = 0$.

$\therefore\ f(0) = f(1)$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist a real number $c \in]0, 1[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow 3c^2 - 4c + 1 = 0 \Rightarrow (c-1)(3c-1) = 0$

$$\Rightarrow c = 1 \text{ or } c = \frac{1}{3}.$$

Out of these two values, clearly $\frac{1}{3} \in]0, 1[$.

Thus, $c = \frac{1}{3} \in]0, 1[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is satisfied.

EXAMPLE 4 *Verify Rolle's theorem for each of the following functions:*

(i) $f(x) = \sin 2x$ *in* $\left[0, \frac{\pi}{2}\right]$

(ii) $f(x) = (\sin x + \cos x)$ *in* $\left[0, \frac{\pi}{2}\right]$

(iii) $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ *in* $\left[0, \frac{\pi}{2}\right]$

(iv) $f(x) = (\sin x - \sin 2x)$ *in* $[0, \pi]$

SOLUTION (i) Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Since the sine function is continuous at each $x \in R$, it follows that $f(x) = \sin 2x$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = 2\cos 2x$, which clearly exists for all $x \in \left[0, \frac{\pi}{2}\right]$.

So, $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[$.

Also, $f(0) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

$$\text{Now,} \quad f'(c) = 0 \Leftrightarrow 2\cos 2c = 0 \Leftrightarrow \cos 2c = 0$$

$$\Leftrightarrow 2c = \frac{\pi}{2}, \text{ i.e., } c = \frac{\pi}{4}.$$

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(ii) Consider $f(x) = (\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]$.

By the continuity of the sine function, the cosine function and the sum of continuous functions, it follows that $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = (\cos x - \sin x)$, which clearly exists for all values of $x \in \left[0, \frac{\pi}{2}\right]$.

So, $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[$. Also, $f(0) = f\left(\frac{\pi}{2}\right) = 1.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \cos c - \sin c = 0 \Leftrightarrow \cos c = \sin c \Leftrightarrow c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(iii) Consider $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ in $\left[0, \frac{\pi}{2}\right]$.

Since the cosine function is continuous everywhere, follows that $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = -2\sin\left(2x - \frac{\pi}{2}\right) = 2\cos 2x$, which clearly exists for all $x \in \left]0, \frac{\pi}{2}\right[$.

$\therefore$ $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[$.

Further, $f(0) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0.$

And, $f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0.$

$\therefore$ $f(0) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow 2\cos 2c = 0 \Leftrightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(iv) Consider $f(x) = (\sin x - \sin 2x)$ in $[0, \pi]$.

Since the sine function is continuous, it follows that $g(x) = \sin x$ and $h(x) = \sin 2x$ are both continuous and so their difference is also continuous.

Consequently, $f(x) = g(x) - h(x)$ is differentiable on $[0, \pi]$.

Also, $f'(x) = (\cos x - 2\cos 2x)$, which clearly exists for all $x \in [0, \pi]$.

$\therefore$ $f(x)$ is differentiable on $]0, \pi[$.

And, $f(0) = f(\pi) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number $c \in]0, \pi[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \cos c - 2\cos 2c = 0$

$$\Leftrightarrow \cos c - 2(2\cos^2 c - 1) = 0$$

$$\Leftrightarrow 4\cos^2 c - \cos c - 2 = 0$$

$$\Leftrightarrow \cos c = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$$

$$\Leftrightarrow \cos c = 0.8431 \text{ or } \cos(180^\circ - c) = 0.5931.$$

$$\Leftrightarrow c = 32^\circ \cdot 32' \text{ or } c = 126^\circ 23'.$$

Thus, $c \in]0, \pi[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is satisfied.

PRACTICE QUESTIONS

Discuss the applicability of Rolle's theorem, when:

$$f(x) = (x-1)(2x-3), \text{ where } 1 \le x \le 3$$

$$f(x) = x^{1/2} \text{ on } [-1, 1]$$

$$f(x) = 2 + (x-1)^{2/3} \text{ on } [0, 2]$$

Using Rolle's theorem, find the point on the curve $y = x(x-4), x \in [0, 4]$, where the tangent is parallel to the x-axis.

ANSWERS

2. Not applicable, since $f(1) \neq f(3)$
3. Not applicable, since $f'(0)$ does not exist
4. Not applicable, since $f'(1)$ does not exist

$$(2, -4)$$

- ***THANK YOU***

Lecture- 40

Lagrange's theorem & its Geometric Interpretations

Lagrange's Mean value Theorem: let $f(x)$ be a real function, such that

(i) $f(x)$ is continuous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b)

Then, there exist atleast one $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Example: Verify Lagrange's mean-value theorem for the function $f(x) = \frac{1}{4x-1}$, $1 \le x \le 4$

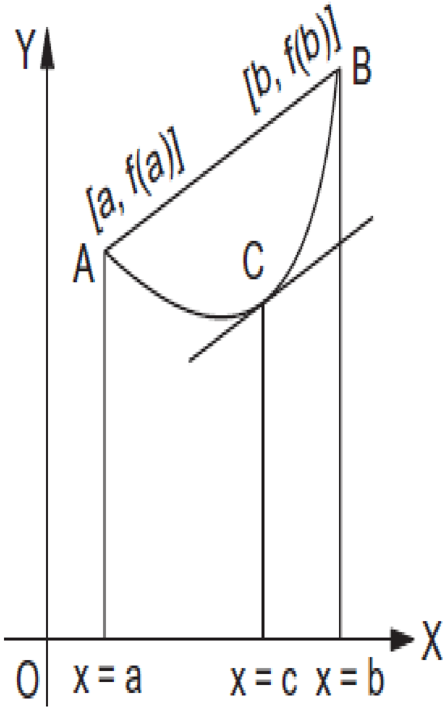
GEOMETRICAL SIGNIFICANCE OF THE MEAN-VALUE THEOREM Let $y = f(x)$ be a given function defined on $[a, b]$, which is continuous on $[a, b]$ and differentiable on $]a, b[$.

Then, by Lagrange's mean-value theorem, there exists some $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{(b - a)} \qquad \dots \text{(i)}$$

Now, if we draw the curve $y = f(x)$ and take the points $A[a, f(a)]$ and $B[b, f(b)]$ on the curve then

$$\text{slope of chord } AB = \frac{f(b) - f(a)}{(b - a)} \qquad \dots \text{(ii)}$$



Thus, from (i) and (ii), we have

$$f'(c) = \text{slope of chord } AB.$$

This shows that *the tangent to the curve $y = f(x)$ at the point $x = c$ is parallel to the chord AB.*

Example: Verify Lagrange's mean-value theorem for the function $f(x) = \frac{1}{4x-1}$, $1 \le x \le 4$

Solution: clearly, for each $x \in [1,4]$, $f(x)$ has a definite & unique value so $f(x)$ is continuous for all $x \in [1,4]$

Also, $f'(x) = \frac{-4}{(4x-1)^2}$, which exist for all $x \in]1,4[$

so $f(x)$ is differentiable on $(1,4)$

→ Both the conditions of Lagrange's mean value theorem are satisfied.

so there must exist some $c \in (1,4)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1}$$

$$f'(c) = \frac{1}{3}\left(\frac{1}{15} - \frac{1}{3}\right) = \frac{-4}{45}$$

now $f'(c) = -\frac{4}{45} \Rightarrow \frac{-4}{(4c-1)^2} = \frac{-4}{45}$

$(4c-1)^2 = 45 \Rightarrow 4c-1 = \pm 3\sqrt{5}$

$c = \frac{1 \pm 3\sqrt{5}}{4}$, clearly $c = \frac{1+3\sqrt{5}}{4} = 1.92$

$= \frac{1 + 3\times 2.23}{4} = 1.92$

lies in $(1,4)$

$\Rightarrow$ Lagrange's mean value theorem is verified.

EXAMPLE *Verify Rolle's theorem for the function* $f(x) = x(x+3)e^{-(x/2)}$ *in* $[-3, 0]$.

EXAMPLE *Verify Rolle's theorem for the function* $f(x) = x(x+3)e^{-(x/2)}$ *in* $[-3, 0]$.

SOLUTION Since a polynomial function as well as an exponential function is continuous and the product of two continuous functions is continuous, it follows that $f(x)$ is continuous on the given interval $[-3, 0]$.

Now, $f'(x) = (2x+3)e^{-(x/2)} - \frac{1}{2}e^{-(x/2)}(x^2+3x)$

$$= e^{-(x/2)}\left(\frac{x+6-x^2}{2}\right),$$

which is clearly finite for all values of x in $]-3, 0[$.

So, $f(x)$ is differentiable on $]-3, 0[$. Also, $f(-3) = f(0) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-3, 0[$ such that $f'(c) = 0$.

But, $f'(c)=0 \Leftrightarrow e^{-(c/2)}\left(\dfrac{c+6-c^2}{2}\right)=0 \Leftrightarrow c+6-c^2=0$

$$\Leftrightarrow (3-c)(c+2)=0 \Leftrightarrow c=3 \text{ or } c=-2.$$

Thus, $c=-2\in\,]-3,0[$ such that $f'(c)=0.$

Hence, Rolle's theorem is verified.

EXAMPLE *Discuss the applicability of Rolle's theorem on:*

$$f(x) = |x| \text{ in } [-1, 1]$$

EXAMPLE *Discuss the applicability of Rolle's theorem on:*

$f(x) = |x|$ *in* $[-1, 1]$

SOLUTION Consider $f(x) = |x|$ in $[-1, 1]$.

We may express it as $f(x) = \begin{cases} -x & \text{when } -1 \le x < 0 \\ x & \text{when } 0 \le x \le 1 \end{cases}$

Clearly, $f(-1) = f(1) = 1$.

But, $Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$

And,

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

$\therefore \; Rf'(0) \ne Lf'(0).$

This shows that $f(x)$ is not differentiable at $x = 0$.

Thus, the condition of differentiability at each point of the given interval is not satisfied.

EXAMPLE *Find a point on the parabola* $y = (x-3)^2$, *where the tangent is parallel to the chord joining* $(3, 0)$ *and* $(4, 1)$.

EXAMPLE *Find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining $(3, 0)$ and $(4, 1)$.*

SOLUTION Let us apply Lagrange's mean-value theorem for the function $f(x) = (x-3)^2$ in the interval $[3, 4]$.

Now, $f(x)$ being a polynomial function, it is continuous on $[3, 4]$.

Also, $f'(x) = 2(x-3)$, which exists for all $x \in]3, 4[$.

So, $f(x)$ is differentiable on $]3, 4[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist a point $c \in]3, 4[$ such that

$$f'(c) = \frac{f(4) - f(3)}{(4-3)} = 1.$$

Now, $f'(c) = 1 \Leftrightarrow 2(c-3) = 1 \Leftrightarrow c = \frac{7}{2} \in]3, 4[$.

Now, $x = \frac{7}{2}$ and $y = (x-3)^2 \Leftrightarrow y = \frac{1}{4}$.

Thus, at the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ on the given curve the tangent is parallel to the chord joining (3, 0) and (4, 1).

EXAMPLE *Verify Lagrange's mean-value theorem for the following functions:*

$$f(x) = x(2-x) \text{ in } [0, 1]$$

EXAMPLE *Verify Lagrange's mean-value theorem for the following functions:*

$f(x) = x(2-x)$ *in* $[0, 1]$

SOLUTION Consider $f(x) = x(2-x)$ in $[0, 1]$.

The given function is $f(x) = 2x - x^2$.

It, being a polynomial function, is continuous on $[0, 1]$.

Also, $f'(x) = 2 - 2x$, which exists for all x in $[0, 1]$.

So, $f(x)$ is differentiable on $]0, 1[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]0, 1[$ such that

$$f'(c) = \frac{f(1) - f(0)}{(1 - 0)} = 1.$$

Now, $f'(c) = 1 \Leftrightarrow 2 - 2c = 1 \Leftrightarrow c = \frac{1}{2} \in]0, 1[.$

Thus, $c = \frac{1}{2} \in]0, 1[$ such that $f'(c) = \frac{f(1) - f(0)}{1 - 0}$.

Hence, Lagrange's mean-value theorem is verified.

Practice Questions

Q 1: Verify *Rolle's* Theorem

$$f(x) = \sqrt{4 - x^2} \text{ in } [-2,\ 2].$$

Ans: Applicable C=0

Q 2: ***Discuss the applicability of Rolle's theorem, when:***

$$f(x) = x^{1/2} \text{ on } [-1, 1]$$

Ans: Not applicable, since $f'(0)$ does not exist

Q 3:

Show that Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$ on $[-1, 1]$.

Q 4:

Find 'c' of Lagrange's mean-value theorem for

(i) $f(x) = (x^3 - 3x^2 + 2x)$ on $\left[0, \frac{1}{2}\right]$

(ii) $f(x) = \sqrt{25 - x^2}$ on $[1, 5]$

Ans: (i) $1 - \sqrt{\frac{7}{12}}$ (ii) $\sqrt{15}$

Thank You

LECTURE NO. -41

Tangent & Normal to a given Curve

Increasing decreasing Function

5. Increasing and Decreasing Functions

INCREASING FUNCTION *A function f(x) defined on]a, b[is said to be increasing if*

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

$$\text{or} \quad x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

STRICTLY INCREASING FUNCTION *A function f(x) defined on]a, b[is said to be strictly increasing if*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

$$\text{or} \quad x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

DECREASING FUNCTION *A function f(x) defined on]a, b[is said to be decreasing if*

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

$$\text{or} \quad x_1 > x_2 \Rightarrow f(x_1) \le f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

STRICTLY DECREASING FUNCTION *A function f(x) defined on]a, b[is said to be strictly decreasing if*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

$$\text{or} \quad x_1 > x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

EXAMPLE 1 *Show that $f(x) = 3x + 5$ is a strictly increasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2$$
$$\Rightarrow 3x_1 + 5 < 3x_2 + 5$$
$$\Rightarrow f(x_1) < f(x_2).$$

Tangents and Normals

<u>THEOREM 1</u> *Prove that the equation of a tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is given by* $\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$,

$\left. dx \right)_{(x_1, y_1)}$

THEOREM 2 *The tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is parallel to the x-axis if and only if* $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$

PROOF The tangent is parallel to the x-axis $\Leftrightarrow$ its slope is 0

$$\Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$$

THEOREM 3 *The tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is parallel to the y-axis if and only if* $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0.$

EXAMPLE 2 *Find the equations of the tangent and the normal to the curve* $y = x^2 + 4x + 1$ *at the point where* $x = 3$.

SOLUTION When $x = 3$, we have $y = (3^2 + 4 \times 3 + 1) = 22$.

So, the point of contact is $(3, 22)$.

Now, $y = x^2 + 4x + 1 \Rightarrow \frac{dy}{dx} = 2x + 4 \Rightarrow \left(\frac{dy}{dx}\right)_{(3,22)} = (2 \times 3 + 4) = 10.$

Equation of the tangent is $\frac{y - 22}{x - 30} = 10 \quad \Rightarrow \quad 10x - y - 278 = 0.$

And, equation of the normal is $\frac{y - 22}{x - 3} = \frac{-1}{10} \quad \Rightarrow \quad x + 10y - 223 = 0.$

EXAMPLE 1 *Find the equations of the tangent and the normal to the curve*

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \text{ at the point } (1, 3).$$

SOLUTION The equation of the given curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10.$$

$$\text{So, } \left(\frac{dy}{dx}\right)_{(1,\,3)} = (4 \times 1^3 - 18 \times 1^2 + 26 \times 1 - 10) = 2.$$

$\therefore$ the required equation of the tangent is

$$\frac{y-3}{x-1} = 2 \text{ or } 2x - y + 1 = 0.$$

And, the required equation of the normal is

$$\frac{y-3}{x-1} = \frac{-1}{2} \text{ or } x + 2y - 7 = 0.$$

1. Derivative as a Rate Measure

Rate of Change of Quantities

Let $y = f(x)$. Then, $\frac{dy}{dx}$ denotes the rate of change of y w.r.t. x and its value at $x = a$ is denoted by $\left[\frac{dy}{dx}\right]_{x=a}$.

If $x = f(t)$, $y = g(t)$ then by chain rule, we have

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right).$$

EXAMPLE 1 *Find the rate of change of the area of a circle with respect to its radius r when $r = 6$ cm.*

SOLUTION Let A be the area of a circle of radius r. Then,

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

$$\Rightarrow \left[\frac{dA}{dr}\right]_{r=6\text{ cm}} = (2\pi \times 6)\text{cm}^2/\text{cm} = (12\pi)\text{cm}^2/\text{cm}.$$

Hence, the area is changing at the rate of (12π) cm^2/cm.

EXAMPLE 2 *A stone is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate of 4 cm/sec, find the rate of increase in its area at the instant when its radius is 10 cm.*

SOLUTION At any instant t, let the radius of the circle be r cm and its area be A cm^2. Then,

$$\frac{dr}{dt} = 4 \text{ cm/sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $A = \pi r^2 \Rightarrow \frac{dA}{dt} = \left(\frac{dA}{dr} \cdot \frac{dr}{dt}\right)$

$$= \frac{d}{dr}(\pi r^2) \cdot 4 \quad \left[\because A = \pi r^2 \text{ and } \frac{dr}{dt} = 4\right]$$

$$= (2\pi r \times 4) \text{ cm}^2/\text{sec} = (8\pi r) \text{ cm}^2/\text{sec}$$

$$\Rightarrow \left[\frac{dA}{dt}\right]_{r=10} = (8\pi \times 10)\ \text{cm}^2/\text{sec} = (80\pi)\ \text{cm}^2/\text{sec}.$$

Hence, the area of the circle is increasing at the rate of $(80\pi)\ \text{cm}^2/\text{sec}$ at the instant when $r = 10$ cm.

GROUP OF INSTITUTIONS

EXAMPLE 3 *A spherical soap bubble is expanding so that its radius is increasing at the rate of* 0.02 cm/sec. *At what rate is the surface area increasing when its radius is* 5 cm? *(Take* $\pi = 3.14$.*)*

SOLUTION A soap bubble is in the form of a sphere. At an instant t, let its radius be r and surface area S. Then,

$$\frac{dr}{dt} = 0.02 \text{ cm/sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $S = 4\pi r^2$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = (8 \times 3.14 \times r \times 0.02) \text{ cm}^2/\text{sec}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{r=5} = (8 \times 3.14 \times 5 \times 0.02)\text{cm}^2/\text{sec} = 2.512 \text{ cm}^2/\text{sec}.$$

Hence, the surface area of the bubble is increasing at the rate of $2.512 \text{ cm}^2/\text{sec}$ at the instance when its radius is 5 cm.

LECTURE NO. -42

Approximation & Errors

2. Errors and Approximation

Let $y = f(x)$. Then, $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x)$.

$$\therefore \quad \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x) + \in, \text{ where } \in \to 0 \text{ when } \delta x \to 0$$

$$\Rightarrow f(x+\delta x)-f(x)=f'(x)\cdot\delta x+\epsilon\cdot\delta x$$

$$\Rightarrow f(x+\delta x)-f(x)=f'(x)\cdot\delta x \quad \text{(approximately)}$$

$$\Rightarrow \delta y=f'(x)\cdot\delta x \quad [\because f(x+\delta x)-f(x)=\delta y].$$

Thus, if δx is an error in x then the corresponding error in y is δy. These small values δx and δy are called *differentials.*

(i) *Absolute Error.* δx is called an absolute error in x.

(ii) *Relative Error.* $\dfrac{\delta x}{x}$ is called the relative error.

(iii) *Percentage Error:* $\left(\dfrac{\delta x}{x}\times 100\right)$ is called the percentage error.

EXAMPLE 1 *Using differentials, find the approximate value of $(82)^{1/4}$ up to three places of decimal.*

SOLUTION Let $f(x) = x^{1/4}$. Then, $f'(x) = \frac{1}{4x^{3/4}}$.

Now, $\{f(x+\delta x) - f(x)\} = f'(x) \cdot \delta x$

$$\Rightarrow \{f(x+\delta x) - f(x)\} = \frac{1}{4x^{3/4}} \cdot \delta x \quad \text{... (i)}$$

We may write, $82 = (81+1)$.

Putting $x = 81$ and $\delta x = 1$ in (i), we get

$$f(81+1) - f(81) = \frac{1}{4 \times (81)^{3/4}} \cdot 1$$

$$\Rightarrow f(82) - f(81) = \frac{1}{(4 \times 3^3)} = \frac{1}{108}$$

$$\Rightarrow f(82) = \left\{f(81) + \frac{1}{108}\right\} = \left\{(81)^{1/4} + \frac{1}{108}\right\} = 3 + 0.009 = 3.009.$$

EXAMPLE 2 *Find the approximate value of the cube root of 127.*

SOLUTION Let $f(x) = x^{1/3}$. Then, $f'(x) = \dfrac{1}{3x^{2/3}}$.

Now, $\{f(x+\delta x) - f(x)\} = f'(x)\cdot \delta x$

$$\Rightarrow \{f(x+\delta x) - f(x)\} = \frac{1}{3x^{2/3}}\cdot \delta x \quad \text{... (i)}$$

We may write, $127 = (125 + 2)$.

Putting $x = 125$ and $\delta x = 2$ in (i), we get

$$f(125+2) - f(125) = \frac{1}{3\times(125)^{2/3}} \times 2$$

$$\Rightarrow \quad f(127) - f(125) = \frac{2}{75}$$

$$\Rightarrow \quad f(127) = f(125) + \frac{2}{75} = \left\{(125)^{1/3} + \frac{2}{75}\right\} = \left(5 + \frac{2}{75}\right) = \frac{377}{75}$$

$$\Rightarrow \quad \sqrt[3]{127} = \frac{377}{75} = 5.026.$$

LECTURE NO. -43,44

MAXIMA & MINIMA OF FUNCTIONS OF ONE VARIABLE

Local Maxima and Local Minima →

let f is a real valued function and let c is an Interior point in the domain of f. then

(i) c is called a point of local maxima if there is an $h > 0$ such that $f(c) > f(x)$ for all x in $(c-h, c+h)$

(ii) c is called a point of local minima if there is an $h > 0$ such that $f(c) < f(x)$ for all x in $(c-h, c+h)$

* The value $f(c)$ is called the local maximum / local minimum value of f.

* The maximum or minimum value of a function is called an extreme or extremum value of function.

* The points at which $f'(x) = 0$ are known as stationary points.

Working Rule to find Extreme Values

I. find $f'(x)$

II. Solve $f'(x)=0$, Each value of x so obtained is the stationary points of f.

III. find $f''(x)$ at each stationary points $x=c$.

if $f''(c)<0$ then $x=c$ is a point of maximum

if $f''(c)>0$ then $x=c$ is a point of minimum

if $f''(c)=0$ then this test fails.

Example → find local maxima or local minima of

$f(x) = x^3 - 6x^2 + 9x + 15$ also find

local maximum or local minimum values.

Sol^n → Given $f(x) = x^3 - 6x^2 + 9x + 15$ —— (1)

$$f'(x) = 3x^2 - 12x + 9$$

Put $f'(x) = 0$ for local maxima and minima

$$3x^2 - 12x + 9 = 0 \Rightarrow 3(x-3)(x-1) = 0$$

$\Rightarrow$ $x = 3$ and $x = 1$ are stationary points

now $f''(x) = 6x - 12$

at $x=3$, $f''(3) = 6\times3-12 = 6 > 0$

So $x=3$ is a point of local minima and local minimum value is given by putting $x=3$ in equation (1)

$$f(3) = (3)^3-6(3)^2+9\times3+15 = 15.$$

at $x=1$ $f''(1) = 6\times1-12 = -6 < 0$

So $x=1$ is a point of local maxima and local maximum value is given by putting $x=1$ in equation (1)

$$f(1) = (1)^3-6(1)^2+9(1)+15 = 19.$$

EXAMPLE *Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function*

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$$

SOLUTION $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$

$\therefore \quad f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15).$

Now, $f'(x) = 0 \Rightarrow -3x(x^2 + 8x + 15) = 0 \Rightarrow -3x(x+5)(x+3) = 0$

$$\Rightarrow x = 0 \text{ or } x = -5 \text{ or } x = -3.$$

Thus, $x = 0$, $x = -5$ and $x = -3$ are the candidates for local maxima or minima.

Moreover, $f''(x) = (-9x^2 - 48x - 45).$

Case I *When* $x = 0$

We have $f''(0) = -45 < 0.$

So, $x = 0$ is a point of local maximum.

And, local maximum value at $x = 0$ is $f(0) = 105.$

Case II *When* $x = -5$

We have $f''(-5) = -30 < 0$.

So, $x = -5$ is a point of local maximum.

And, local maximum value at $x = -5$ is $f(-5) = \dfrac{295}{4}$.

Case III *When* $x = -3$

We have $f''(-3) = 18 > 0$.

So, $x = -3$ is a point of local minimum.

Local minimum value at $x = -3$ is $f(-3) = \dfrac{231}{4}$.

EXAMPLE *Find the local maxima and local minima of the functions:*

$$f(x) = (\sin 2x - x), \text{ where } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

EXAMPLE *Find the local maxima and local minima of the functions:*

$$f(x) = (\sin 2x - x), \textit{ where } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

SOLUTION $f(x) = (\sin 2x - x) \Rightarrow f'(x) = (2\cos 2x - 1)$ and $f''(x) = -4\sin 2x$.

$$\therefore\ f'(x) = 0 \Rightarrow (2\cos 2x - 1) = 0 \Rightarrow \cos 2x = \frac{1}{2}$$

$$\Rightarrow 2x = -\frac{\pi}{3} \text{ or } 2x = \frac{\pi}{3} \Rightarrow x = \frac{-\pi}{6} \text{ or } x = \frac{\pi}{6}.$$

Thus, $x = \frac{-\pi}{6}$ and $x = \frac{\pi}{6}$ are the candidates for local maxima or local minima.

Case I *When* $x = -(\pi/6)$

We have $f''\left(-\frac{\pi}{6}\right) = 4\sin\frac{\pi}{3} = 2\sqrt{3} > 0.$

So, $x = \frac{-\pi}{6}$ is a point of local minimum.

The local minimum value

$$= f\left(-\frac{\pi}{6}\right) = \left(-\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = -\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right).$$

Case II $\quad$ *When* $x = \frac{\pi}{6}$

We have $f''\left(\frac{\pi}{6}\right) = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3} < 0$

$\therefore \quad x = \frac{\pi}{6}$ is a point of local maximum.

And, the local maximum value

$$= f\left(\frac{\pi}{6}\right) = \left(\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right).$$

EXAMPLE *Find the points of local maxima or local minima of the function*

$$f(x) = (\sin^4 x + \cos^4 x) \;\; in \;\; 0 < x < \frac{\pi}{2}.$$

EXAMPLE *Find the points of local maxima or local minima of the function*

$$f(x) = (\sin^4 x + \cos^4 x) \;\; in \;\; 0 < x < \frac{\pi}{2}.$$

SOLUTION $f(x) = \sin^4 x + \cos^4 x$

$$\Rightarrow f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -4\sin x \cos x(\cos^2 x - \sin^2 x) = -2\sin 2x \cos 2x = -\sin 4x.$$

And, $f''(x) = -4\cos 4x.$

Now, $f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow 4x = \pi, \text{ i.e., } x = \frac{\pi}{4}.$

$\therefore \;\; x = (\pi/4)$ is a point of local maximum or local minimum.

Now, $f''\left(\frac{\pi}{4}\right) = -4\cos \pi = 4 > 0.$

$\therefore \;\; x = (\pi/4)$ is a point of local minimum.

Local minimum value $= f\left(\frac{\pi}{4}\right) = \frac{1}{2}.$

PRACTICE QUESTIONS

1. Find the stationary point of the function $y = x^2 - 2x + 3$ and hence determine the .nature of this point.

2. Find the stationary points of the functions (a) f(x) = $3x^2 + 2x - 9$, (b) f(x) = $x^3 - 6x^2 + 9x - 2$

3. Find the stationary points of the function $y = 2x^3 - 9x^2 + 12x - 3$ and determine their nature.

Answers

1. Local minima at (1,2)

2. a) $(-1/3, -28/3)$.

   b) $(3, -2)$ and $(1, 2)$.

3. local maximum point at (1,2) and the local minimum point at (2,1)

THANK YOU