

# ■ UNIT-5

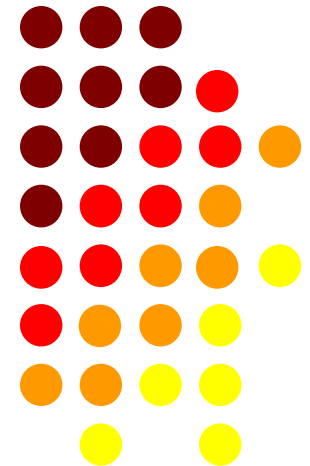
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# Lecture- 39



## Rolls theorem & its Geometric Interpretations



## Rolle's Theorem

Let  $f$  be a real-valued function, defined in the closed interval  $[a, b]$  such that

(i)  $f$  is continuous on  $[a, b]$ ;      (ii)  $f$  is differentiable on  $]a, b[$ ;

(iii)  $f(a) = f(b)$ .

Then, there exists a real number  $c$  in the open interval  $]a, b[$  such that

$$f'(c) = 0$$



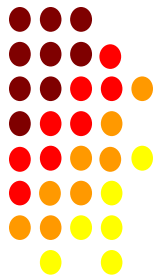
**GEOMETRICAL SIGNIFICANCE OF ROLLE'S THEOREM** Let  $f$  be a real function defined on  $[a, b]$  and let Rolle's theorem be applicable on it. Then,  $f$  being continuous on  $[a, b]$ , it follows that we can draw a graph of  $f(x)$  between the values  $x = a$  and  $x = b$ .

Also,  $f(x)$  being differentiable in  $]a, b[$ , it follows that the graph of  $f(x)$  has a tangent at each point of  $]a, b[$ .

Now, the existence of a real number  $c \in ]a, b[$  such that  $f'(c) = 0$  shows that the tangent to the curve at  $x = c$  has a slope 0, i.e., it is parallel to the  $x$ -axis.



EXAMPLE 1 *Verify Rolle's theorem for the function  $f(x) = x^3 - 6x^2 + 11x - 6$  in the interval  $[1, 3]$ .*



SOLUTION Here, we observe that

(i)  $f(x)$  being a polynomial function of  $x$ , is continuous on the interval  $]1, 3]$ .

(ii)  $f'(x) = 3x^2 - 12x + 11$ , which clearly exists for all values of  $x \in ]1, 3]$ .

So,  $f(x)$  is differentiable on the open interval  $]1, 3[$ .

(iii)  $f(1) = (1^3 - 6 \times 1^2 + 11 \times 1 - 6) = 0$

and  $f(3) = (3^3 - 6 \times 3^2 + 11 \times 3 - 6) = 0$

$\therefore f(1) = f(3)$ .

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some  $c \in ]1, 3[$  such that  $f'(c) = 0$ .

Now,  $f'(c) = 0 \Rightarrow 3c^2 - 12c + 11 = 0$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 132}}{6} \Rightarrow c = \left(2 \pm \frac{1}{\sqrt{3}}\right).$$

Clearly, both the values of  $c$  lie in the interval  $]1, 3[$ .

Hence, Rolle's theorem is verified.



EXAMPLE 2 Verify Rolle's theorem for the function  $f(x) = x(x - 1)^2$  in the interval  $[0, 1]$ .



SOLUTION We have  $f(x) = x^3 - 2x^2 + x$ .

We observe here that

(i)  $f(x)$  being a polynomial function, is continuous on  $]0, 1[$ .

(ii)  $f'(x) = (3x^2 - 4x + 1)$ , which clearly exists for all values of  $x \in ]0, 1[$ .

So,  $f(x)$  is differentiable on the interval  $]0, 1[$ .

(iii)  $f(0) = 0$  and  $f(1) = 0$ .

$\therefore f(0) = f(1)$ .

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist a real number  $c \in ]0, 1[$  such that  $f'(c) = 0$

Now,  $f'(c) = 0 \Rightarrow 3c^2 - 4c + 1 = 0 \Rightarrow (c-1)(3c-1) = 0$

$$\Rightarrow c = 1 \text{ or } c = \frac{1}{3}.$$

Out of these two values, clearly  $\frac{1}{3} \in ]0, 1[$ .

Thus,  $c = \frac{1}{3} \in ]0, 1[$  such that  $f'(c) = 0$ .

Hence, Rolle's theorem is satisfied.



EXAMPLE 4 *Verify Rolle's theorem for each of the following functions:*

(i)  $f(x) = \sin 2x$  in  $\left[0, \frac{\pi}{2}\right]$

(ii)  $f(x) = (\sin x + \cos x)$  in  $\left[0, \frac{\pi}{2}\right]$

(iii)  $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$  in  $\left[0, \frac{\pi}{2}\right]$

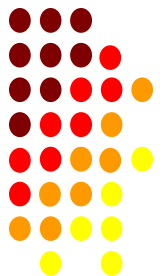
(iv)  $f(x) = (\sin x - \sin 2x)$  in  $[0, \pi]$



SOLUTION (i) Consider  $f(x) = \sin 2x$  in  $\left[0, \frac{\pi}{2}\right]$ .

Since the sine function is continuous at each  $x \in \mathbb{R}$ , it follows that  $f(x) = \sin 2x$  is continuous on  $\left[0, \frac{\pi}{2}\right]$ .

Also,  $f'(x) = 2 \cos 2x$ , which clearly exists for all  $x \in \left[0, \frac{\pi}{2}\right]$ .



So,  $f(x)$  is differentiable on  $\left]0, \frac{\pi}{2}\right[$ .

$$\text{Also, } f(0) = f\left(\frac{\pi}{2}\right) = 0.$$

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist a real number  $c \in \left]0, \frac{\pi}{2}\right[$  such that  $f'(c) = 0$ .

$$\begin{aligned} \text{Now, } f'(c) = 0 &\Leftrightarrow 2 \cos 2c = 0 \Leftrightarrow \cos 2c = 0 \\ &\Leftrightarrow 2c = \frac{\pi}{2}, \text{ i.e., } c = \frac{\pi}{4}. \end{aligned}$$

Thus,  $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$  such that  $f'(c) = 0$ .

Hence, Rolle's theorem is verified.



(ii) Consider  $f(x) = (\sin x + \cos x)$  in  $\left[0, \frac{\pi}{2}\right]$ .

By the continuity of the sine function, the cosine function and the sum of continuous functions, it follows that  $f(x)$  is continuous on  $\left[0, \frac{\pi}{2}\right]$ .

Also,  $f'(x) = (\cos x - \sin x)$ , which clearly exists for all values of  $x \in \left[0, \frac{\pi}{2}\right]$ .

So,  $f(x)$  is differentiable on  $\left]0, \frac{\pi}{2}\right[$ . Also,  $f(0) = f\left(\frac{\pi}{2}\right) = 1$ .

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some  $c \in \left]0, \frac{\pi}{2}\right[$  such that  $f'(c) = 0$ .

Now,  $f'(c) = 0 \Leftrightarrow \cos c - \sin c = 0 \Leftrightarrow \cos c = \sin c \Leftrightarrow c = \frac{\pi}{4}$ .

Thus,  $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$  such that  $f'(c) = 0$ .

Hence, Rolle's theorem is verified.



(iii) Consider  $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$  in  $\left[0, \frac{\pi}{2}\right]$ .

Since the cosine function is continuous everywhere,

follows that  $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$  is continuous on  $\left[0, \frac{\pi}{2}\right]$ .



Also,  $f'(x) = -2 \sin\left(2x - \frac{\pi}{2}\right) = 2 \cos 2x$ , which clearly exists for all  $x \in \left]0, \frac{\pi}{2}\right[$ .

$\therefore f(x)$  is differentiable on  $\left]0, \frac{\pi}{2}\right[$ .

Further,  $f(0) = \cos 2\left(-\frac{\pi}{4}\right) = \cos \frac{\pi}{2} = 0$ .

And,  $f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos \frac{\pi}{2} = 0$

$\therefore f(0) = f\left(\frac{\pi}{2}\right) = 0$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist  $c \in \left]0, \frac{\pi}{2}\right[$  such that  $f'(c) = 0$ .

Now,  $f'(c) = 0 \Leftrightarrow 2 \cos 2c = 0 \Leftrightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4}$ .

Thus,  $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$  such that  $f'(c) = 0$

Hence, Rolle's theorem is verified.



(iv) Consider  $f(x) = (\sin x - \sin 2x)$  in  $[0, \pi]$ .

Since the sine function is continuous, it follows that  $g(x) = \sin x$  and  $h(x) = \sin 2x$  are both continuous and so their difference is also continuous.

Consequently,  $f(x) = g(x) - h(x)$  is differentiable on  $[0, \pi]$ .

Also,  $f'(x) = (\cos x - 2 \cos 2x)$ , which clearly exists for all  $x \in [0, \pi]$ .

$\therefore f(x)$  is differentiable on  $]0, \pi[$ .

And,  $f(0) = f(\pi) = 0$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number  $c \in ]0, \pi[$  such that  $f'(c) = 0$ .

Now,  $f'(c) = 0 \Leftrightarrow \cos c - 2 \cos 2c = 0$

$$\Leftrightarrow \cos c - 2(2 \cos^2 c - 1) = 0$$

$$\Leftrightarrow 4 \cos^2 c - \cos c - 2 = 0$$

$$\Leftrightarrow \cos c = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$$

$$\Leftrightarrow \cos c = 0.8431 \text{ or } \cos(180^\circ - c) = 0.5931.$$

$$\Leftrightarrow c = 32^\circ 32' \text{ or } c = 126^\circ 23'.$$

Thus,  $c \in ]0, \pi[$  such that  $f'(c) = 0$ .

Hence, Rolle's theorem is satisfied.



# PRACTICE QUESTIONS

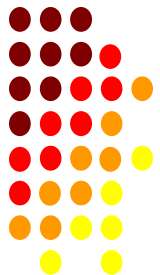
*Discuss the applicability of Rolle's theorem, when:*

$$f(x) = (x - 1)(2x - 3), \text{ where } 1 \leq x \leq 3$$

$$f(x) = x^{1/2} \text{ on } [-1, 1]$$

$$f(x) = 2 + (x - 1)^{2/3} \text{ on } [0, 2]$$

Using Rolle's theorem, find the point on the curve  $y = x(x - 4)$ ,  $x \in [0, 4]$ , where the tangent is parallel to the  $x$ -axis.



# ANSWERS

2. Not applicable, since  $f(1) \neq f(3)$
3. Not applicable, since  $f'(0)$  does not exist
4. Not applicable, since  $f'(1)$  does not exist

$$(2, -4)$$



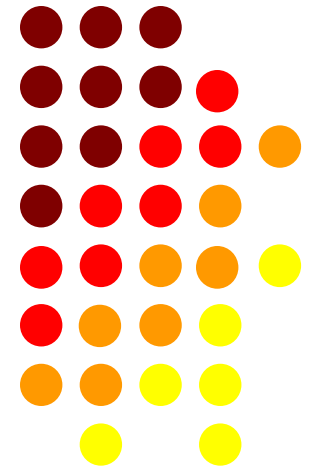
**THANK YOU**



# Lecture- 40



## Lagrange's theorem & its Geometric Interpretations



Lagrange's Mean Value Theorem : Let  $f(x)$  be a real function, such that

(i)  $f(x)$  is continuous on  $[a, b]$

(ii)  $f(x)$  is differentiable on  $(a, b)$

Then, there exist atleast one  $c \in (a, b)$  such

that 
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Example: Verify Lagrange's mean-value theorem  
for the function  $f(x) = \frac{1}{4x-1}$ ,  $1 \leq x \leq 4$ .



**GEOMETRICAL SIGNIFICANCE OF THE MEAN-VALUE THEOREM** Let  $y = f(x)$  be a given function defined on  $[a, b]$ , which is continuous on  $[a, b]$  and differentiable on  $]a, b[$ .

Then, by Lagrange's mean-value theorem, there exists some  $c \in ]a, b[$  such that

$$f'(c) = \frac{f(b) - f(a)}{(b - a)} \quad \dots (i)$$

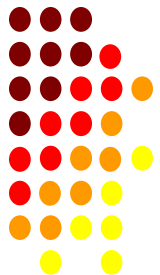
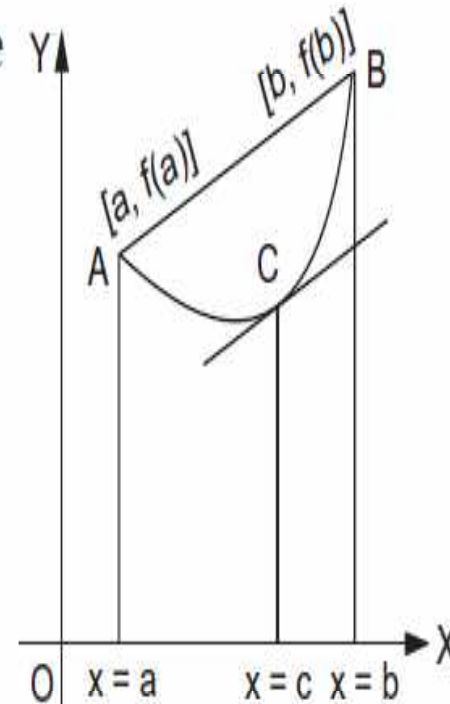


Now, if we draw the curve  $y = f(x)$  and take the points  $A[a, f(a)]$  and  $B[b, f(b)]$  on the curve then

$$\text{slope of chord } AB = \frac{f(b) - f(a)}{(b - a)} \quad \dots \text{ (ii)}$$

Thus, from (i) and (ii), we have  
 $f'(c) = \text{slope of chord } AB.$

This shows that *the tangent to the curve  $y = f(x)$  at the point  $x = c$  is parallel to the chord  $AB$ .*



Example: Verify Lagrange's mean-value theorem for the function  $f(x) = \frac{1}{4x-1}$ ,  $1 \leq x \leq 4$

Solution: clearly, for each  $x \in [1, 4]$ ,  $f(x)$  has a definite & unique value so  $f(x)$  is continuous for all  $x \in [1, 4]$

Also,  $f'(x) = \frac{-4}{(4x-1)^2}$ , which exist for all  $x \in ]1, 4[$

so  $f(x)$  is differentiable on  $(1, 4)$

$\Rightarrow$  Both the conditions of Lagrange's mean value theorem are satisfied.

so there must exist some  $c \in (1, 4)$  such

$$\text{that } f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1}$$

$$f'(c) = \frac{1}{3} \left( \frac{1}{15} - \frac{1}{3} \right) = \frac{-4}{45}$$



$$\text{now } f'(c) = -\frac{4}{45} \Rightarrow \frac{-4}{(4c-1)^2} = -\frac{4}{45}$$

$$(4c-1)^2 = 45 \Rightarrow 4c-1 = \pm 3\sqrt{5}$$

$$c = \frac{1 \pm 3\sqrt{5}}{4}, \text{ clearly } c = \frac{1+3\sqrt{5}}{4} = 1.92$$
$$= \frac{1+3 \times 2.23}{4} = 1.92$$

lies in  $(1,4)$

$\Rightarrow$  Lagrange's mean value theorem is verified.



EXAMPLE Verify Rolle's theorem for the function  $f(x) = x(x + 3)e^{-(x/2)}$  in  $[-3, 0]$ .



EXAMPLE Verify Rolle's theorem for the function  $f(x) = x(x + 3)e^{-(x/2)}$  in  $[-3, 0]$ .

SOLUTION Since a polynomial function as well as an exponential function is continuous and the product of two continuous functions is continuous, it follows that  $f(x)$  is continuous on the given interval  $[-3, 0]$ .

$$\begin{aligned}\text{Now, } f'(x) &= (2x + 3)e^{-(x/2)} - \frac{1}{2}e^{-(x/2)}(x^2 + 3x) \\ &= e^{-(x/2)} \left( \frac{x + 6 - x^2}{2} \right),\end{aligned}$$

which is clearly finite for all values of  $x$  in  $] -3, 0[$ .

So,  $f(x)$  is differentiable on  $] -3, 0[$ . Also,  $f(-3) = f(0) = 0$ .

Thus, all the conditions of Rolle's theorem are satisfied.

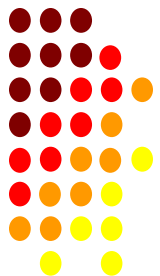
So, there must exist  $c \in ] -3, 0[$  such that  $f'(c) = 0$ .



$$\begin{aligned} \text{But, } f'(c) = 0 &\Leftrightarrow e^{-(c/2)} \left( \frac{c+6-c^2}{2} \right) = 0 \Leftrightarrow c+6-c^2 = 0 \\ &\Leftrightarrow (3-c)(c+2) = 0 \Leftrightarrow c = 3 \text{ or } c = -2. \end{aligned}$$

Thus,  $c = -2 \in ]-3, 0[$  such that  $f'(c) = 0$ .

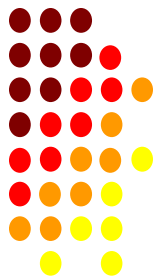
Hence, Rolle's theorem is verified.



EXAMPLE

*Discuss the applicability of Rolle's theorem on:*

$$f(x) = |x| \text{ in } [-1, 1]$$



EXAMPLE Discuss the applicability of Rolle's theorem on:

$$f(x) = |x| \text{ in } [-1, 1]$$

SOLUTION

Consider  $f(x) = |x|$  in  $[-1, 1]$ .

$$\text{We may express it as } f(x) = \begin{cases} -x & \text{when } -1 \leq x < 0 \\ x & \text{when } 0 \leq x \leq 1 \end{cases}$$

Clearly,  $f(-1) = f(1) = 1$ .

$$\text{But, } Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

And,

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{|-h|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1.$$

$$\therefore Rf'(0) \neq Lf'(0).$$

This shows that  $f(x)$  is not differentiable at  $x = 0$ .

Thus, the condition of differentiability at each point of the given interval is not satisfied.



EXAMPLE Find a point on the parabola  $y = (x - 3)^2$ , where the tangent is parallel to the chord joining  $(3, 0)$  and  $(4, 1)$ .



EXAMPLE

Find a point on the parabola  $y = (x - 3)^2$ , where the tangent is parallel to the chord joining  $(3, 0)$  and  $(4, 1)$ .

SOLUTION

Let us apply Lagrange's mean-value theorem for the function  $f(x) = (x - 3)^2$  in the interval  $[3, 4]$ .

Now,  $f(x)$  being a polynomial function, it is continuous on  $[3, 4]$ .

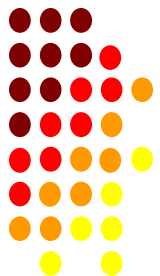
Also,  $f'(x) = 2(x - 3)$ , which exists for all  $x \in ]3, 4[$ .

So,  $f(x)$  is differentiable on  $]3, 4[$ .

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist a point  $c \in ]3, 4[$  such that

$$f'(c) = \frac{f(4) - f(3)}{(4 - 3)} = 1.$$



$$\text{Now, } f'(c) = 1 \Leftrightarrow 2(c-3) = 1 \Leftrightarrow c = \frac{7}{2} \in ]3, 4[.$$

$$\text{Now, } x = \frac{7}{2} \text{ and } y = (x-3)^2 \Leftrightarrow y = \frac{1}{4}.$$

Thus, at the point  $\left(\frac{7}{2}, \frac{1}{4}\right)$  on the given curve the tangent is parallel to the chord joining  $(3, 0)$  and  $(4, 1)$ .



EXAMPLE

*Verify Lagrange's mean-value theorem for the following functions:*

$$f(x) = x(2-x) \text{ in } [0, 1]$$



EXAMPLE

Verify Lagrange's mean-value theorem for the following functions:

$$f(x) = x(2 - x) \text{ in } [0, 1]$$

SOLUTION

Consider  $f(x) = x(2 - x)$  in  $[0, 1]$ .

The given function is  $f(x) = 2x - x^2$ .

It, being a polynomial function, is continuous on  $[0, 1]$ .

Also,  $f'(x) = 2 - 2x$ , which exists for all  $x$  in  $[0, 1]$ .

So,  $f(x)$  is differentiable on  $]0, 1[$ .

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.



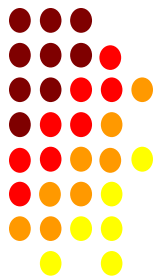
So, there must exist some  $c \in ]0, 1[$  such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 1.$$

Now,  $f'(c) = 1 \Leftrightarrow 2 - 2c = 1 \Leftrightarrow c = \frac{1}{2} \in ]0, 1[$ .

Thus,  $c = \frac{1}{2} \in ]0, 1[$  such that  $f'(c) = \frac{f(1) - f(0)}{1 - 0}$ .

Hence, Lagrange's mean value theorem is verified.



# Practice Questions

Q 1: Verify *Rolle's* Theorem

$$f(x) = \sqrt{4 - x^2} \text{ in } [-2, 2].$$

**Ans:** Applicable  $C=0$

Q 2: *Discuss the applicability of Rolle's theorem, when:*

$$f(x) = x^{1/2} \text{ on } [-1, 1]$$

**Ans:** Not applicable, since  $f'(0)$  does not exist



Q 3:

Show that Lagrange's mean-value theorem is not applicable to  $f(x) = \frac{1}{x}$  on  $[-1, 1]$ .

Q 4:

Find 'c' of Lagrange's mean-value theorem for

(i)  $f(x) = (x^3 - 3x^2 + 2x)$  on  $\left[0, \frac{1}{2}\right]$

(ii)  $f(x) = \sqrt{25 - x^2}$  on  $[1, 5]$

**Ans:** (i)  $1 - \sqrt{\frac{7}{12}}$  (ii)  $\sqrt{15}$



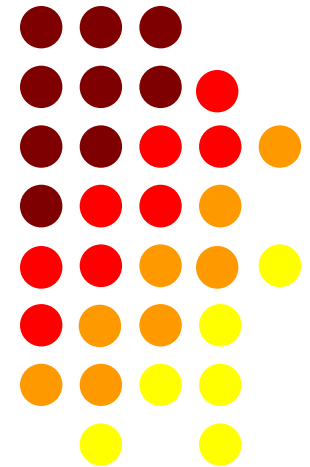
# Thank You



**LECTURE NO. -41**



# **Tangent & Normal to a given Curve**



# Increasing decreasing Function

## 5. Increasing and Decreasing Functions

**INCREASING FUNCTION** A function  $f(x)$  defined on  $]a, b[$  is said to be increasing if

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \text{ for all } x_1, x_2 \in ]a, b[$$

or  $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$  for all  $x_1, x_2 \in ]a, b[$ .

**STRICTLY INCREASING FUNCTION** A function  $f(x)$  defined on  $]a, b[$  is said to be strictly increasing if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in ]a, b[$$

or  $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$  for all  $x_1, x_2 \in ]a, b[$ .

**DECREASING FUNCTION** A function  $f(x)$  defined on  $]a, b[$  is said to be decreasing if

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \text{ for all } x_1, x_2 \in ]a, b[$$

or  $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$  for all  $x_1, x_2 \in ]a, b[$ .

**STRICTLY DECREASING FUNCTION** A function  $f(x)$  defined on  $]a, b[$  is said to be strictly decreasing if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in ]a, b[$$

or  $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$  for all  $x_1, x_2 \in ]a, b[$ .



**EXAMPLE 1** Show that  $f(x) = 3x + 5$  is a strictly increasing function on  $\mathbb{R}$ .

**SOLUTION** Let  $x_1, x_2 \in \mathbb{R}$  such that  $x_1 < x_2$ . Then,

$$\begin{aligned}x_1 < x_2 &\Rightarrow 3x_1 < 3x_2 \\ &\Rightarrow 3x_1 + 5 < 3x_2 + 5 \\ &\Rightarrow f(x_1) < f(x_2).\end{aligned}$$



# Tangents and Normals

THEOREM 1 Prove that the equation of a tangent to a curve  $y = f(x)$  at a point

$$P(x_1, y_1) \text{ is given by } \frac{y - y_1}{x - x_1} = \left( \frac{dy}{dx} \right)_{(x_1, y_1)}$$



THEOREM 2 The tangent to a curve  $y = f(x)$  at a point  $P(x_1, y_1)$  is parallel to the  $x$ -axis if and only if  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$ .

PROOF The tangent is parallel to the  $x$ -axis  $\Leftrightarrow$  its slope is 0  
 $\Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$ .

THEOREM 3 The tangent to a curve  $y = f(x)$  at a point  $P(x_1, y_1)$  is parallel to the  $y$ -axis if and only if  $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$ .



**EXAMPLE 2** Find the equations of the tangent and the normal to the curve  $y = x^2 + 4x + 1$  at the point where  $x = 3$ .

**SOLUTION** When  $x = 3$ , we have  $y = (3^2 + 4 \times 3 + 1) = 22$ .

So, the point of contact is  $(3, 22)$ .

$$\text{Now, } y = x^2 + 4x + 1 \Rightarrow \frac{dy}{dx} = 2x + 4 \Rightarrow \left(\frac{dy}{dx}\right)_{(3, 22)} = (2 \times 3 + 4) = 10.$$

$$\text{Equation of the tangent is } \frac{y - 22}{x - 3} = 10 \Rightarrow 10x - y - 278 = 0.$$

$$\text{And, equation of the normal is } \frac{y - 22}{x - 3} = \frac{-1}{10} \Rightarrow x + 10y - 223 = 0.$$



EXAMPLE 1 Find the equations of the tangent and the normal to the curve

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \text{ at the point } (1, 3).$$

**SOLUTION** The equation of the given curve is  $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ .

$$\therefore \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10.$$

$$\text{So, } \left( \frac{dy}{dx} \right)_{(1, 3)} = (4 \times 1^3 - 18 \times 1^2 + 26 \times 1 - 10) = 2.$$

$\therefore$  the required equation of the tangent is

$$\frac{y - 3}{x - 1} = 2 \text{ or } 2x - y + 1 = 0.$$

And, the required equation of the normal is

$$\frac{y - 3}{x - 1} = \frac{-1}{2} \text{ or } x + 2y - 7 = 0.$$







## 1. Derivative as a Rate Measure

### Rate of Change of Quantities

Let  $y = f(x)$ . Then,  $\frac{dy}{dx}$  denotes the rate of change of  $y$  w.r.t.  $x$  and its value at  $x = a$  is denoted by  $\left[ \frac{dy}{dx} \right]_{x=a}$ .

If  $x = f(t)$ ,  $y = g(t)$  then by chain rule, we have

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left( \frac{dy}{dt} \cdot \frac{dt}{dx} \right).$$



EXAMPLE 1 Find the rate of change of the area of a circle with respect to its radius  $r$  when  $r = 6$  cm.

**SOLUTION** Let  $A$  be the area of a circle of radius  $r$ . Then,

$$\begin{aligned} A &= \pi r^2 \Rightarrow \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r \\ &\Rightarrow \left[ \frac{dA}{dr} \right]_{r=6 \text{ cm}} = (2\pi \times 6) \text{ cm}^2 / \text{cm} = (12\pi) \text{ cm}^2 / \text{cm}. \end{aligned}$$

Hence, the area is changing at the rate of  $(12\pi) \text{ cm}^2 / \text{cm}$ .



**EXAMPLE 2** *A stone is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate of 4 cm/sec, find the rate of increase in its area at the instant when its radius is 10 cm.*

**SOLUTION** At any instant  $t$ , let the radius of the circle be  $r$  cm and its area be  $A$  cm<sup>2</sup>. Then,

$$\frac{dr}{dt} = 4 \text{ cm/sec} \quad (\text{given}) \quad \dots \text{ (i)}$$

$$\begin{aligned} \text{Now, } A = \pi r^2 \Rightarrow \frac{dA}{dt} &= \left( \frac{dA}{dr} \cdot \frac{dr}{dt} \right) \\ &= \frac{d}{dr} (\pi r^2) \cdot 4 \quad \left[ \because A = \pi r^2 \text{ and } \frac{dr}{dt} = 4 \right] \\ &= (2\pi r \times 4) \text{ cm}^2/\text{sec} = (8\pi r) \text{ cm}^2/\text{sec} \end{aligned}$$



$$\Rightarrow \left[ \frac{dA}{dt} \right]_{r=10} = (8\pi \times 10) \text{ cm}^2/\text{sec} = (80\pi) \text{ cm}^2/\text{sec}.$$

Hence, the area of the circle is increasing at the rate of  $(80\pi) \text{ cm}^2/\text{sec}$  at the instant when  $r = 10 \text{ cm}$ .



**EXAMPLE 3** *A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area increasing when its radius is 5 cm? (Take  $\pi = 3.14$ .)*

**SOLUTION** A soap bubble is in the form of a sphere. At an instant  $t$ , let its radius be  $r$  and surface area  $S$ . Then,

$$\frac{dr}{dt} = 0.02 \text{ cm/sec (given)} \quad \dots \text{ (i)}$$

$$\text{Now, } S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = (8 \times 3.14 \times r \times 0.02) \text{ cm}^2/\text{sec}$$

$$\Rightarrow \left[ \frac{dS}{dt} \right]_{r=5} = (8 \times 3.14 \times 5 \times 0.02) \text{ cm}^2/\text{sec} = 2.512 \text{ cm}^2/\text{sec}.$$

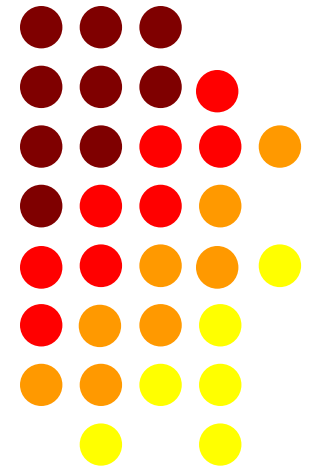
Hence, the surface area of the bubble is increasing at the rate of  $2.512 \text{ cm}^2/\text{sec}$  at the instance when its radius is 5 cm.



**LECTURE NO. -42**



# **Approximation & Errors**



## 2. Errors and Approximation

Let  $y = f(x)$ . Then,  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x)$ .

$\therefore \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x) + \epsilon$ , where  $\epsilon \rightarrow 0$  when  $\delta x \rightarrow 0$



$$\Rightarrow f(x + \delta x) - f(x) = f'(x) \cdot \delta x + \epsilon \cdot \delta x$$

$$\Rightarrow f(x + \delta x) - f(x) = f'(x) \cdot \delta x \quad (\text{approximately})$$

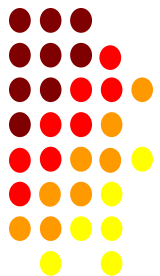
$$\Rightarrow \delta y = f'(x) \cdot \delta x \quad [ \because f(x + \delta x) - f(x) = \delta y ].$$

Thus, if  $\delta x$  is an error in  $x$  then the corresponding error in  $y$  is  $\delta y$ . These small values  $\delta x$  and  $\delta y$  are called *differentials*.

(i) *Absolute Error*.  $\delta x$  is called an absolute error in  $x$ .

(ii) *Relative Error*.  $\frac{\delta x}{x}$  is called the relative error.

(iii) *Percentage Error*.  $\left( \frac{\delta x}{x} \times 100 \right)$  is called the percentage error.



**EXAMPLE 1** Using differentials, find the approximate value of  $(82)^{1/4}$  up to three places of decimal.

**SOLUTION** Let  $f(x) = x^{1/4}$ . Then,  $f'(x) = \frac{1}{4x^{3/4}}$ .

Now,  $\{f(x + \delta x) - f(x)\} = f'(x) \cdot \delta x$

$$\Rightarrow \{f(x + \delta x) - f(x)\} = \frac{1}{4x^{3/4}} \cdot \delta x \quad \dots (i)$$

We may write,  $82 = (81 + 1)$ .

Putting  $x = 81$  and  $\delta x = 1$  in (i), we get

$$f(81 + 1) - f(81) = \frac{1}{4 \times (81)^{3/4}} \cdot 1$$

$$\Rightarrow f(82) - f(81) = \frac{1}{(4 \times 3^3)} = \frac{1}{108}$$

$$\Rightarrow f(82) = \left\{ f(81) + \frac{1}{108} \right\} = \left\{ (81)^{1/4} + \frac{1}{108} \right\} = 3 + 0.009 = 3.009.$$



EXAMPLE 2 Find the approximate value of the cube root of 127.

SOLUTION Let  $f(x) = x^{1/3}$ . Then,  $f'(x) = \frac{1}{3x^{2/3}}$ .

Now,  $\{f(x + \delta x) - f(x)\} = f'(x) \cdot \delta x$

$$\Rightarrow \{f(x + \delta x) - f(x)\} = \frac{1}{3x^{2/3}} \cdot \delta x \quad \dots (i)$$

We may write,  $127 = (125 + 2)$ .

Putting  $x = 125$  and  $\delta x = 2$  in (i), we get

$$f(125 + 2) - f(125) = \frac{1}{3 \times (125)^{2/3}} \times 2$$



$$\Rightarrow f(127) - f(125) = \frac{2}{75}$$

$$\Rightarrow f(127) = f(125) + \frac{2}{75} = \left\{ (125)^{\frac{1}{3}} + \frac{2}{75} \right\} = \left( 5 + \frac{2}{75} \right) = \frac{377}{75}$$

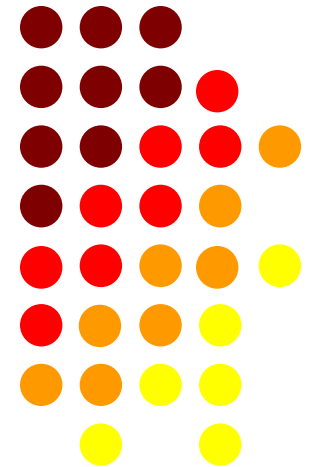
$$\Rightarrow \sqrt[3]{127} = \frac{377}{75} = 5.026.$$



**LECTURE NO. -43,44**



**MAXIMA & MINIMA  
OF FUNCTIONS OF  
ONE VARIABLE**



## Local Maxima and Local Minima →

Let  $f$  is a real valued function and let  $c$  is an interior point in the domain of  $f$ . Then

(i)  $c$  is called a point of local maxima if there is an  $h > 0$  such that  $f(c) > f(x)$  for all  $x$  in  $(c-h, c+h)$

(ii)  $c$  is called a point of local minima if there is an  $h > 0$  such that  $f(c) < f(x)$  for all  $x$  in  $(c-h, c+h)$

\* The value  $f(c)$  is called the local maximum / local minimum value of  $f$ .

\* The maximum or minimum value of a function is called an extreme or extremum value of function.

\* The points at which  $f'(x) = 0$  are known as stationary points.



## Working Rule to find Extreme Values

- I. find  $f'(x)$
- II. Solve  $f'(x) = 0$  , Each value of  $x$  so obtained is the stationary points of  $f$ .
- III. find  $f''(x)$  at each stationary points  $x = c$ .
  - if  $f''(c) < 0$  then  $x = c$  is a point of maximum
  - if  $f''(c) > 0$  then  $x = c$  is a point of minimum
  - if  $f''(c) = 0$  then this test fails.



Example → find local maxima or local minima of

$$f(x) = x^3 - 6x^2 + 9x + 15 \quad \text{also find}$$

local maximum or local minimum values.

Sol<sup>n</sup> → Given  $f(x) = x^3 - 6x^2 + 9x + 15$  — ①

$$f'(x) = 3x^2 - 12x + 9$$

Put  $f'(x) = 0$  for local maxima and minima

$$3x^2 - 12x + 9 = 0 \Rightarrow 3(x-3)(x-1) = 0$$

$\Rightarrow x = 3$  and  $x = 1$  are stationary points

now  $f''(x) = 6x - 12$



at  $x=3$ ,  $f''(3) = 6 \times 3 - 12 = 6 > 0$

So  $x=3$  is a point of local minima  
and local minimum value is given by putting  
 $x=3$  in equation (1)

$$f(3) = (3)^3 - 6(3)^2 + 9 \times 3 + 15 = \underline{\underline{15}}$$

at  $x=1$   $f''(1) = 6 \times 1 - 12 = -6 < 0$

So  $x=1$  is a point of local maxima,  
and local maximum value is given by putting  
 $x=1$  in equation (1)

$$f(1) = (1)^3 - 6(1)^2 + 9(1) + 15 = \underline{\underline{19}}$$



EXAMPLE

Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$$

SOLUTION

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$$

$$\therefore f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15).$$

$$\begin{aligned} \text{Now, } f'(x) = 0 &\Rightarrow -3x(x^2 + 8x + 15) = 0 \Rightarrow -3x(x + 5)(x + 3) = 0 \\ &\Rightarrow x = 0 \text{ or } x = -5 \text{ or } x = -3. \end{aligned}$$

Thus,  $x = 0$ ,  $x = -5$  and  $x = -3$  are the candidates for local maxima or minima.

$$\text{Moreover, } f''(x) = (-9x^2 - 48x - 45).$$

Case I When  $x = 0$

$$\text{We have } f''(0) = -45 < 0.$$

So,  $x = 0$  is a point of local maximum.

And, local maximum value at  $x = 0$  is  $f(0) = 105$ .



Case II When  $x = -5$

We have  $f''(-5) = -30 < 0$

So,  $x = -5$  is a point of local maximum.

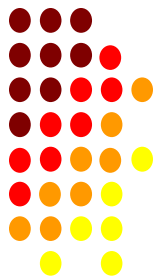
And, local maximum value at  $x = -5$  is  $f(-5) = \frac{295}{4}$ .

Case III When  $x = -3$

We have  $f''(-3) = 18 > 0$

So,  $x = -3$  is a point of local minimum.

Local minimum value at  $x = -3$  is  $f(-3) = \frac{231}{4}$ .



**EXAMPLE**

*Find the local maxima and local minima of the functions:*

$$f(x) = (\sin 2x - x), \text{ where } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$



**EXAMPLE** Find the local maxima and local minima of the functions:

$$f(x) = (\sin 2x - x), \text{ where } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

**SOLUTION**  $f(x) = (\sin 2x - x) \Rightarrow f'(x) = (2\cos 2x - 1)$  and  $f''(x) = -4\sin 2x$ .

$$\therefore f'(x) = 0 \Rightarrow (2\cos 2x - 1) = 0 \Rightarrow \cos 2x = \frac{1}{2}$$

$$\Rightarrow 2x = -\frac{\pi}{3} \text{ or } 2x = \frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6} \text{ or } x = \frac{\pi}{6}.$$

Thus,  $x = -\frac{\pi}{6}$  and  $x = \frac{\pi}{6}$  are the candidates for local maxima or local minima.

Case I When  $x = -(\pi/6)$

$$\text{We have } f''\left(-\frac{\pi}{6}\right) = 4\sin \frac{\pi}{3} = 2\sqrt{3} > 0.$$

So,  $x = -\frac{\pi}{6}$  is a point of local minimum.

The local minimum value

$$= f\left(-\frac{\pi}{6}\right) = \left(-\sin \frac{\pi}{3} - \frac{\pi}{6}\right) = -\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right).$$



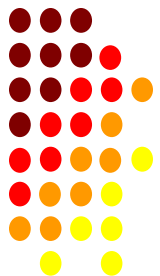
Case II When  $x = \frac{\pi}{6}$

$$\text{We have } f''\left(\frac{\pi}{6}\right) = -4 \sin\left(\frac{\pi}{3}\right) = -2\sqrt{3} < 0$$

$\therefore x = \frac{\pi}{6}$  is a point of local maximum.

And, the local maximum value

$$= f\left(\frac{\pi}{6}\right) = \left(\sin \frac{\pi}{3} - \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right).$$



EXAMPLE

*Find the points of local maxima or local minima of the function*

$$f(x) = (\sin^4 x + \cos^4 x) \text{ in } 0 < x < \frac{\pi}{2}$$



EXAMPLE

Find the points of local maxima or local minima of the function

$$f(x) = (\sin^4 x + \cos^4 x) \text{ in } 0 < x < \frac{\pi}{2}.$$

SOLUTION

$$f(x) = \sin^4 x + \cos^4 x$$

$$\Rightarrow f'(x) = 4 \sin^3 x \cos x - 4 \cos^3 x \sin x$$

$$= -4 \sin x \cos x (\cos^2 x - \sin^2 x) = -2 \sin 2x \cos 2x = -\sin 4x.$$

$$\text{And, } f''(x) = -4 \cos 4x.$$

$$\text{Now, } f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow 4x = \pi, \text{ i.e., } x = \frac{\pi}{4}.$$

$\therefore x = (\pi/4)$  is a point of local maximum or local minimum.

$$\text{Now, } f''\left(\frac{\pi}{4}\right) = -4 \cos \pi = 4 > 0.$$

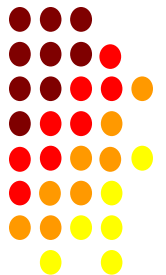
$\therefore x = (\pi/4)$  is a point of local minimum.

$$\text{Local minimum value} = f\left(\frac{\pi}{4}\right) = \frac{1}{2}.$$



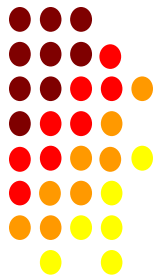
# PRACTICE QUESTIONS

1. Find the stationary point of the function  $y = x^2 - 2x + 3$  and hence determine the nature of this point.
2. Find the stationary points of the functions (a)  $f(x) = 3x^2 + 2x - 9$ , (b)  $f(x) = x^3 - 6x^2 + 9x - 2$
3. Find the stationary points of the function  $y = 2x^3 - 9x^2 + 12x - 3$  and determine their nature.



# Answers

1. Local minima at  $(1,2)$
2. a )  $(-1/3, -28/3)$ .  
b)  $(3, -2)$  and  $(1, 2)$ .
3. local maximum point at  $(1,2)$  and the local minimum point at  $(2,1)$



**THANK YOU**

