Unit IV
(Multivariate Calculus-I)
L-27

Introduction to double integrals

Introduction:→ In this unit we will study about double & triple integrals, which are very useful in finding area, volume, mass, centroid etc.

Double Integral :→ An integral of the form

$$J = \iint_R f(x,y)\,dx\,dy$$

is called double integral of f(x,y) over the region R, which can also be written as

$$J = \iint_R f(x,y)\,dy\,dx.$$

Evaluation of Double Integral ⇒ The method of evaluating the double integrals depend upon the nature of the curves bounding the region R.

case-a) ⇒ If the region R is defined as ⇒

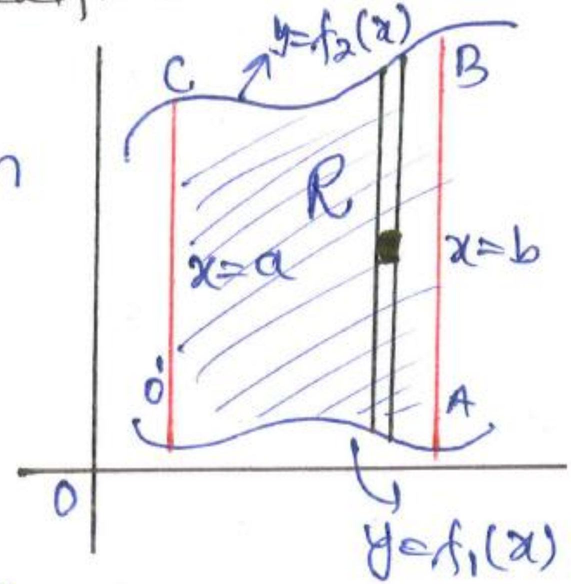
$a \le x \le b$, $f_1(x) \le y \le f_2(y)$

Here region R is the region represented by O'ABC, first we will take a vertical strip & treating x as a constant. Then we have lower limit of x is a

lower limit of y is $f_1(x)$

upper limit of x is b

& upper limit of y is $f_2(x)$.



Now the double integral is evaluated first w.r.t y (treating x as a constant) & then w.r.t x, then we have

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx$$

Ques → Evaluate $\int_0^a \int_0^x xy \, dy \, dx$ [2013-14]

Sol^n → here $[0 = f_1(x)] \le y \; [x = f_2(x)]$

& $[0 \le x \le a]$

$$\text{So } I = \int_{x=0}^{a} \int_{y=0}^{x} xy \, dy \, dx$$

$$= \int_{x=0}^{a} x \left[\int_{y=0}^{x} y \, dy\right] dx = \int_{x=0}^{a} x \left(\frac{y^2}{2}\right)_0^x dx$$

$$= \int_{x=0}^{a} x \cdot \left[\frac{x^2}{2} - 0\right] dx = \frac{1}{2} \int_{x=0}^{a} x^3 \, dx$$

$$= \frac{1}{2} \left(\frac{x^4}{4}\right)_0^a = \frac{a^4}{8} \quad \text{Ans.}$$

Ques: Evaluate $\int_0^a \int_{x/a}^{x} \frac{x}{x^2+y^2}\,dy\,dx$

Sol: $I=\int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{x/a}^{x} dx$ $\qquad \left[\because \int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

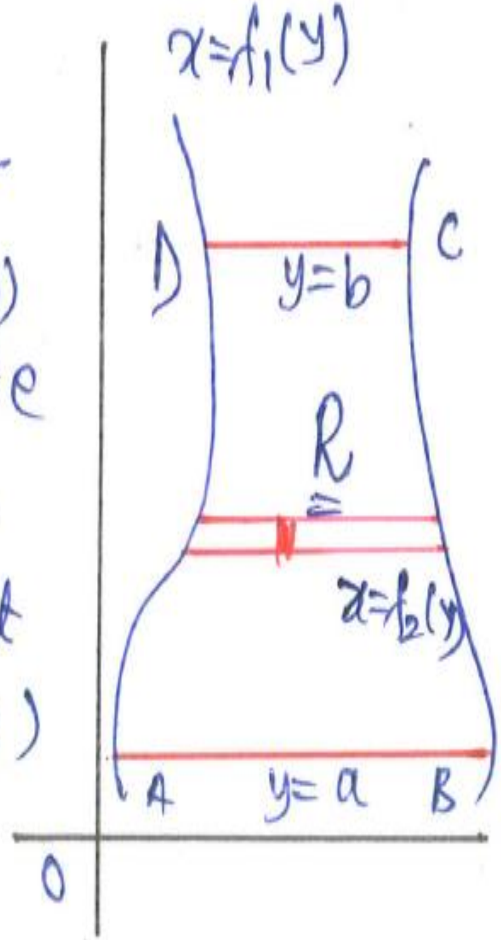
$$=\int_0^a x\left[\frac{1}{x}\tan^{-1}1-\frac{1}{x}\tan^{-1}\frac{1}{a}\right]dx=\int_0^a\left(\frac{\pi}{4}-\tan^{-1}\frac{1}{a}\right)dx$$

$$=\left(\frac{\pi}{4}x-x\tan^{-1}\frac{1}{a}\right)_0^a=\frac{\pi}{4}a-a\tan^{-1}\frac{1}{a}$$ **Ans**

Case: b) ⇒ If Region R is defined as

$a \le y \le b, \quad f_1(y) \le x \le f_2(y)$

Here region R is bounded by the boundaries $y=a, y=b, x=f_1(y)$ & $x=f_2(y)$, Now to integrate we take a horizontal strip in R & integrate first w.r.t x (by keeping y as a constant) & then integrate w.r.t



y, so the integral becomes

$$\iint_R f(x,y)\,dx\,dy = \int_{y=a}^{y=b} \left[\int_{x=f_1(y)}^{f_2(y)} f(x,y)\, dx \right] dy$$

Ques: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

Sol^n: Here $0 \le y \le a$ & $0 \le x \le \sqrt{a^2-y^2}$ so we integrate first w.r.t x.

$$I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy$$

$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

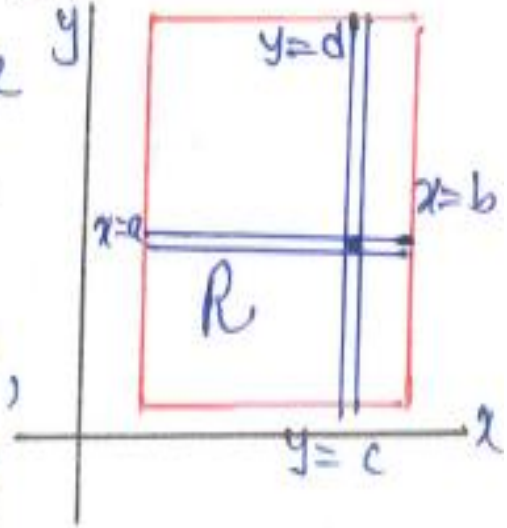
$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}}$$

$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1\, dy = \frac{\pi}{4} \int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right]$$

$$= \frac{\pi a^3}{6} \quad \text{Ans.}$$

Case: c ⇒ If the Region R is defined as $a \le x \le b$ & $c \le y \le d$ (where a, b, c & d are constants).
In this case the order of integration is immaterial, provided that the limits of integrations are changed accordingly.



Ques: Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

Sol^n: Here $0 \le x \le 1$ & $0 \le y \le 1$ so both variables have constant limits.

$$I = \int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\int_0^1 \frac{1}{\sqrt{1-y^2}} dy \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\sin^{-1} y \right]_0^1 dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\frac{\pi}{2} - 0 \right] dx$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \left(\sin^{-1} x \right)_0^1 = \frac{\pi}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$ Ans.

Ques: Evaluate $\int_1^2 \int_3^4 (xy+e^y)\,dy\,dx = \int_1^2 \left[\int_3^4 (xy+e^y)\,dy\right]dx$

$$= \int_1^2 \left[\frac{xy^2}{2}+e^y\right]_3^4 dx = \int_1^2 \left(8x+e^4-\frac{9}{2}x-e^3\right)dx$$

$$= \int_1^2 \left(\frac{7}{2}x+e^4-e^3\right)dx = \left[\frac{7x^2}{4}+\left(e^4-e^3\right)x\right]_1^2$$

$$= 7+2\left(e^4-e^3\right)-\frac{7}{4}-\left(e^4-e^3\right)$$

$$= \frac{21}{4}+e^4-e^3 \quad \text{Ans}$$

Exp: Evaluate $\int_0^2 \int_0^1 (x^2+3y^2)\,dy\,dx$ [2019-20]

Sol^n: $I = \int_0^2 \left[\int_0^1 (x^2+3y^2)\,dy\right]dx = \int_0^2 \left[x^2 y + \frac{3y^3}{3}\right]_0^1 dx$

$= \int_0^2 \left[x^2+1\right]dx = \left(\frac{x^3}{3}+x\right)_0^2 = \frac{8}{3}+2$

$= \frac{8+6}{3} = \frac{14}{3}$ Ans.

Exp: Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} \, dx \, dy$ [2018-19]

Sol^n: Given $I = \int_0^1 \int_0^{x^2} e^{y/x} \, dy \, dx$

here $0 \le y \le x^2$ & $0 \le x \le 1$

$$I = \int_0^1 \left(\frac{e^{y/x}}{1/x} \right)_0^{x^2} dx = \int_0^1 x \left(e^{y/x} \right)_0^{x^2} dx$$

$$= \int_0^1 x \left[e^x - 1 \right] dx$$

$$= \int_0^1 \left[x e^x - x \right] dx = \left[x e^x - \int 1 \times e^x \, dx - \frac{x^2}{2} \right]_0^1$$

$$= \left[x e^x - e^x - \frac{x^2}{2} \right]_0^1$$

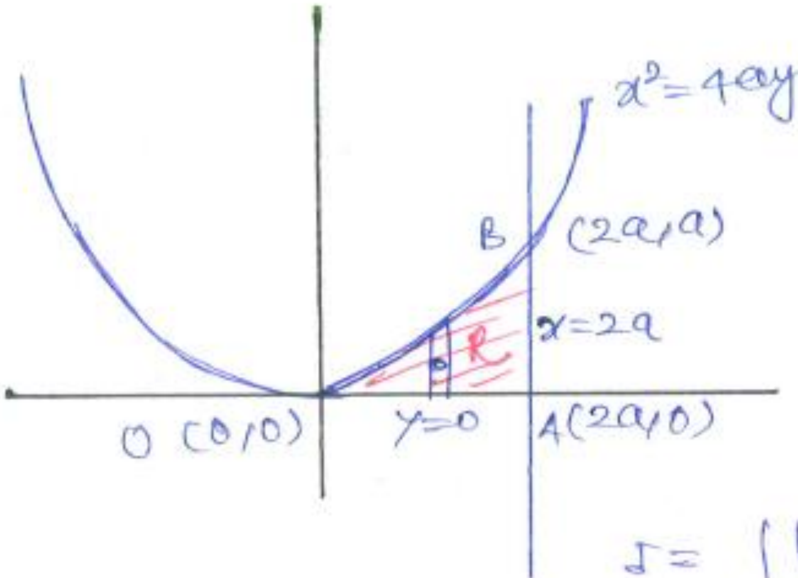
$$= \frac{1}{2}$$

$e^0 = 1$ & $e^1 = e$

Exp: Find the value of the integral $\iint xy\,dx\,dy$, where R is the region bounded by the x-axis, the line $x=2a$ & the parabola $x^2=4ay$.

Sol: Here region R is the common part bounded by x-axis, $x=2a$ & $x^2=4ay$

[2011-12]



Intersection of $x^2=4ay$ & $x=2a$ is (2a, a).

R is the region OAB, which is given by

$0 \le x \le 2a$ &

$0 \le y \le \frac{x^2}{4a}$

so Given integral is

$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{\frac{x^2}{4a}} xy\,dy\,dx$$

$$= \int_{x=0}^{2a} x\left(\frac{y^2}{2}\right)_0^{\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x^5}{32a^2}\,dx$$

$$= \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{a^4}{3} \quad \text{Ans.}$$

Exp: Let D be the region in the first quadrant bounded by the curves $xy = 16$, $x = y$, $y = 0$ & $x = 8$. Sketch the region of integration of the following integral $\iint_D x^2\,dx\,dy$ & evaluate it.

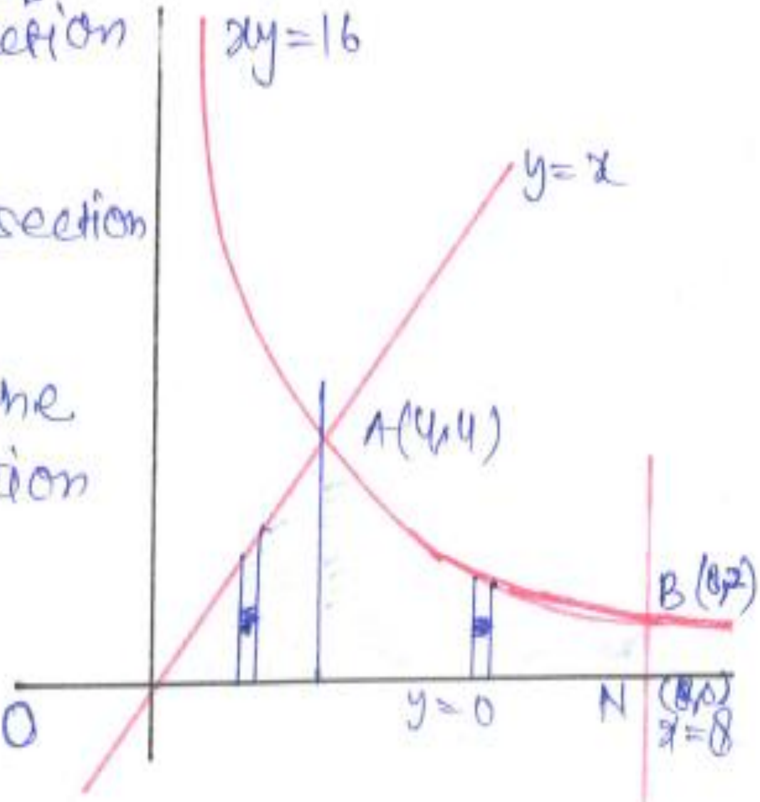
Sol^n: A(4, 4) is the intersection of $xy = 16$ & $y = x$.

B(8, 2) is the intersection of $xy = 16$ & $x = 8$.

Shaded portion is the region of integration.

Given

$$I = \iint_D x^2\,dx\,dy$$



To evaluate the given integral, we divide the area OABNO into two parts by AM as shown in the figure.

so $I = \iint_D x^2 \, dx \, dy = \iint_{OMAO} x^2 \, dx \, dy + \iint_{MNBAM} x^2 \, dx \, dy$

$= \int_{x=0}^{4} \int_{y=0}^{x} x^2 \, dy \, dx + \int_{x=4}^{8} \int_{y=0}^{\frac{16}{x}} x^2 \, dy \, dx$

$= \int_0^4 x^2 (y)_0^x \, dx + \int_4^8 x^2 (y)_0^{16/x} \, dx$

$= \int_0^4 x^3 \, dx + \int_4^8 16x \, dx = \left(\frac{x^4}{4}\right)_0^4 + \left(8x^2\right)_4^8$

$= 64 + 8(64-16)$

$= 64 + 384$

$= 448$ Ans.

Ques No Evaluate $\int_0^a \int_{x/a}^{x} \frac{x}{x^2+y^2}\,dydx$

Sol: $I=\int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{x/a}^{x} dx$ $\left[\because \int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$$=\int_0^a x\left[\frac{1}{x}\tan^{-1}1-\frac{1}{x}\tan^{-1}\frac{1}{a}\right]dx=\int_0^a\left(\frac{\pi}{4}-\tan^{-1}\frac{1}{a}\right)dx$$

$$=\left(\frac{\pi}{4}x-x\tan^{-1}\frac{1}{a}\right)_0^a=\frac{\pi}{4}a-a\tan^{-1}\frac{1}{a}$$ **Ans**

Ques No **Evaluate** $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

Sol. $I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\,dx \right] dy$

$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1\,dy = \frac{\pi}{4}\int_0^a (a^2-y^2)\,dy = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$= \frac{\pi a^3}{6}$ **Ans**

Practice Question

1. Evaluate $\int_0^1 \int_0^{x^2} x e^y \, dy \, dx$ [2017-18]

2. Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

3. Evaluate $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}} \, dy \, dx$

Ans 1:

Ans 2: $\frac{\pi^2}{4}$

Ans 3: $e^2 - 1$

Practice Question

1. Evaulate: $\iint_R xy\,dx\,dy$ where R is the domain bounded by $x-axis, ordiante\ x = 2a$ and the curve $x^2 = 4ay$

2. Evaulate: $\iint_R xy\,dx\,dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0,\ y \geq 0$

3. Evaluate $\int_1^0 \int_0^1 (x+y)\,dx\,dy$

4. Evaluate $\int_0^1 \int_0^x e^x\,dx\,dy$

Ans 1: $\frac{a^4}{3}$

Ans 2: $\frac{a^4}{8}$

Ans 3: 1

Ans 4: 1

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Double integral in Polar Coordinates

Introduction: Double integrals of the form

$$I = \int_{\theta=\theta_1}^{\theta_2} \int_{r=r_1}^{r_2} f(r,\theta)\, dr\, d\theta$$

is known as double integral in Polar coordinates. To solve these types of integrals we first solve w.r.t. r between the limits r_1 & r_2 & then solve w.r.t θ between θ_1 to θ_2.

Exp: Evaluate $I = \int_0^{2\pi} \int_{a\sin\theta}^{a} r\,dr\,d\theta$.

Sol.

$$I = \int_0^{2\pi} \int_{r=a\sin\theta}^{a} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{a\sin\theta}^{a} d\theta = \frac{1}{2}\int_0^{2\pi} \left(a^2 - a^2\sin^2\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{a^2}{2}\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$I = \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{\pi a^2}{2}.$$

Exp: Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$.

Sol: we have $I = \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$.

$$= \int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{a^3}{3} - \frac{a^3(1-\cos\theta)^3}{3}\right] d\theta.$$

$$= \frac{a^3}{3}\int_{\theta=0}^{\pi/2}\left[1-(1-\cos\theta)^3\right] d\theta.$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[1-(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\right] d\theta.$$

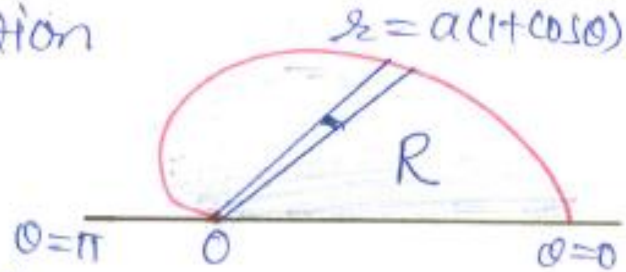
$$= \frac{a^3}{3}\int_0^{\pi/2}\left[3\cos\theta - 3\cos^2\theta + \cos^3\theta\right] d\theta.$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[3\cos\theta - 3\left(\frac{1+\cos 2\theta}{2}\right) + \frac{1}{4}(\cos 3\theta + 3\cos\theta\right] d\theta.$$

$$= \frac{a^3}{3}\left[3\sin\theta - \frac{3}{2}\theta - \frac{3}{2}\frac{\sin 2\theta}{2} + \frac{1}{4}\frac{\sin 3\theta}{3} + 3\sin\theta\right]_0^{\pi/2} = \frac{a^3}{36}\left[44-9\pi\right] \text{ Ans}$$

Exp: Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ above the initial line.

Soln: Region 'R' of integration is given by



$0 \le \theta \le \pi$

& $0 \le r \le a(1+\cos\theta)$.

$$I = \iint_R r\sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{0=r}^{a(1+\cos\theta)} r\sin\theta \, dr\, d\theta.$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta.$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} \sin\theta \; a^2(1+\cos\theta)^2 \, d\theta.$$

putting $1+\cos\theta = t \Rightarrow \sin\theta\, d\theta = -dt$

$\theta = 0 \Rightarrow t = 2$

$\theta = \pi \Rightarrow t = 0$

$$I = \frac{1}{2}\int_2^0 a^2\, t^2\,(-dt) = \frac{a^2}{2}\int_0^2 t^2\, dt$$

$$= \frac{a^2}{2}\left(\frac{t^3}{3}\right)_0^2 = \frac{a^2}{2} \times \frac{8}{3}$$

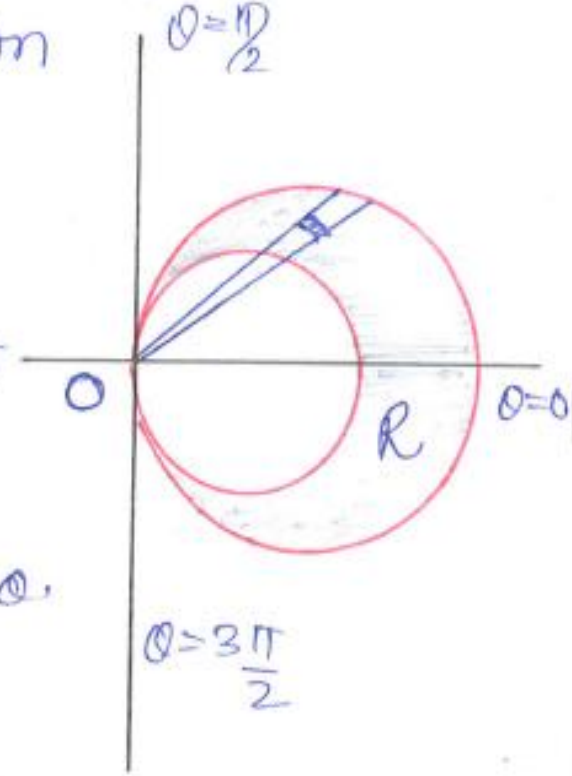
$$= \frac{4a^2}{3}$$ Ans

Exp: Evaluate $\iint r^3 \, dr \, d\theta$, over the area bounded between the circles $r = 2\cos\theta$ & $r = 4\cos\theta$.

Sol^n: Region R of integration is given by

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ &

$2\cos\theta \le r \le 4\cos\theta$;



$$I = \iint_R r^3 \, dr \, d\theta.$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=2\cos\theta}^{4\cos\theta} r^3 \, dr \, d\theta.$$

$$I = \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4} \right)_{2\cos\theta}^{4\cos\theta} d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}\left[256\cos^4\theta - 16\cos^4\theta\right] d\theta.$$

$$= 120 \int_{0}^{\pi/2} \cos^4\theta \, d\theta.$$

$$= 120 \times \frac{3\times 1}{4\times 2} \times \frac{\pi}{2} = \frac{45}{2}\pi$$

formula $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \frac{1\cdot 3\cdot 5 \cdots (n-1)}{2\cdot 4\cdot 6 \cdots n} \cdot \frac{\pi}{2}$

If n is even

$\displaystyle\int_0^{\pi/2} \cos^n\theta \, d\theta = \int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{2\cdot 4\cdot 6 \cdots (n-1)}{1\cdot 3\cdot 5 \cdots n}$, If n is odd

Practice Question

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Evaluate the following:

1. $\iint r e^{\frac{-r^2}{a^2}} \cos\theta \sin\theta \, dr \, d\theta$, over the upper half of the circle $r = 2a\cos\theta$.

$$\left[\text{Ans.}: \frac{a^2}{16}\left(3 + \frac{1}{e^4}\right)\right]$$

2. $\iint r^3 dr \, d\theta$, over the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$$\left[\text{Ans.}: \frac{45\pi}{2}\right]$$

3. $\iint r \sin\theta \, dA$, over the cardioid $r = a(1 + \cos\theta)$ above the initial line.

$$\left[\text{Ans.}: \frac{4}{3}a^3\right]$$

4. $\iint \frac{r}{\sqrt{r^2 + 4}} dr \, d\theta$, over one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$$[\text{Ans.}: (4 - \pi)]$$

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Change of order of integration

Introduction: On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration. value of the integration is unchanged by changing of the order of integration.

Some complicated integrals can be made easy to handle by a change in the order of integration.

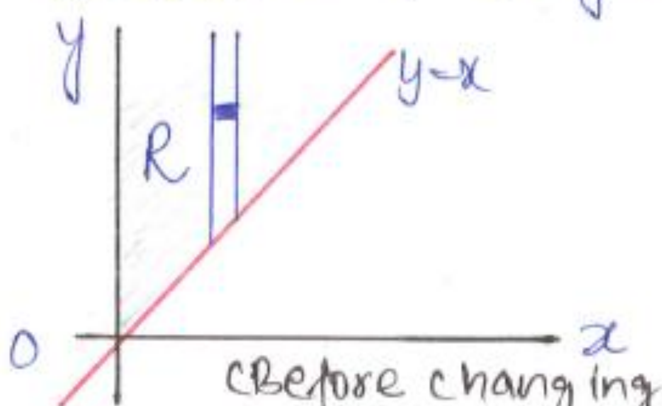
Exp: Evaluate the following integral by changing the order of integration, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

[2020-21]

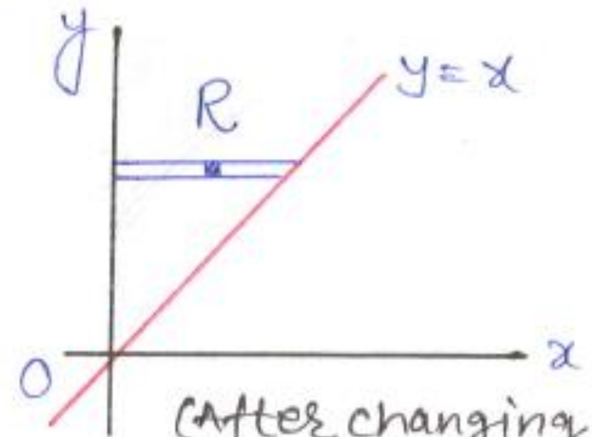
Sol^n: Given integral is $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Given region of integration is

$0 \le x \le \infty$ & $x \le y \le \infty$



(Before changing the order)



(After changing the order)

After changing of the order of integration R can be written as $0 \le x \le y$, $0 \le y < \infty$

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So $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ (Before)

$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$ (After)

$= \int_0^\infty \left(\frac{e^{-y}}{y}\right) \int_0^y (1)\, dx\, dy$

$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y\, dy = \int_0^\infty \frac{e^{-y}}{y} \times y\, dy$

$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = \left(-\frac{1}{e^y}\right)_0^\infty = -\left[\frac{1}{e^\infty} - 1\right] = -(0-1)$

$= 1$ Ans.

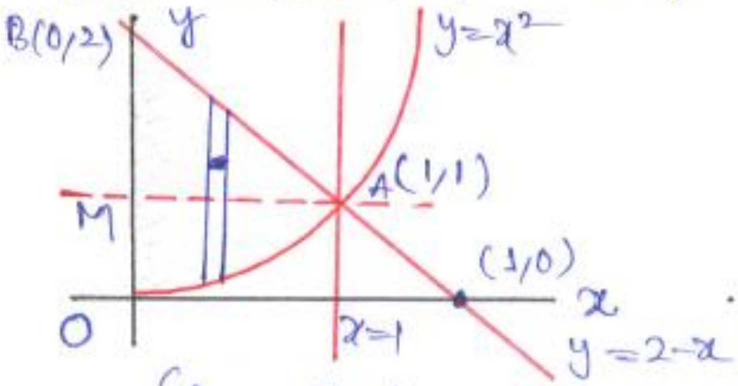
Exp: Evaluate the integration by changing of order of $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

[2014-15, 2015-16, 2016-17, 2017-18, 2019-20]

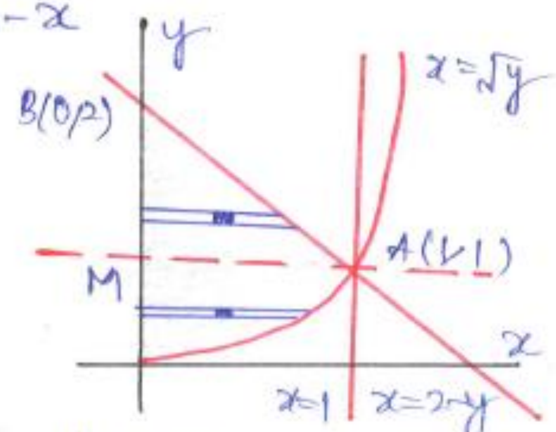
Soln: Given $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx \quad -(1)$

where region of integral is bounded by the curves $y = x^2$, $y = 2-x$, $x=0$ & $x=1$

i.e $0 \le x \le 1$, $x^2 \le y \le 2-x$



Given Region



After changing the order

After changing the order we have

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OAMO} xy\,dx\,dy + \iint_{MABM} xy\,dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \frac{1}{2} \int_0^1 y\,(x^2)_0^{\sqrt{y}}\,dy + \int_1^2 y \left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2} \int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y\,(2-y)^2\,dy$$

$$= \frac{1}{2} \left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2} \int_1^2 \left[4y - 4y^2 + y^3\right] dy$$

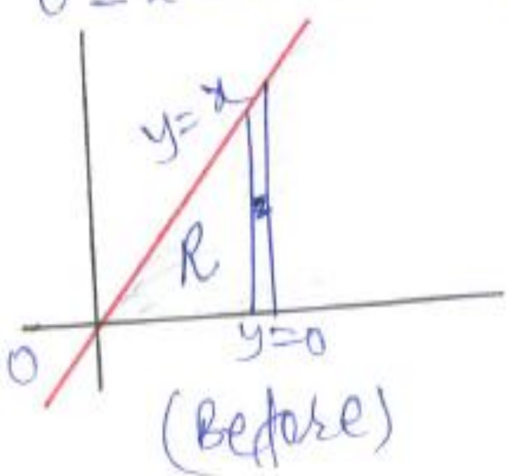
$$= \frac{1}{6} + \frac{1}{2} \left[2y^3 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

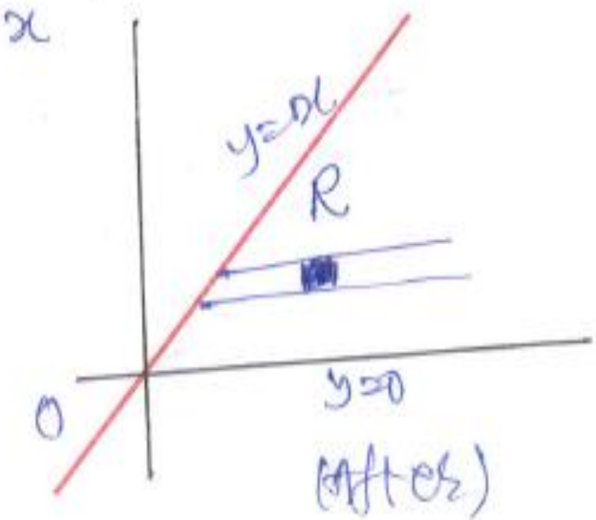
$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{3}{8} \quad \text{Ans.}$$

Exp: Evaluate the integral $\int_0^{\infty}\int_0^{x} x\cdot\exp\left(-\frac{x^2}{y}\right)dx\,dy$, by changing the order of integration.

Soln: Region of Given integral is bounded by $0 \le x < \infty$, $0 \le y \le x$





After changing the order, region is given by

$y \leq x < \infty$, $0 \leq y < \infty$

$$I = \int_0^\infty \int_0^x x \cdot \exp\left(-\frac{x^2}{y}\right) dy\, dx \quad \text{(Before)}$$

$$= \int_0^\infty \int_y^\infty x \cdot e^{-(x^2/y)}\, dx\, dy \quad \text{(After)}$$

$$= \int_0^\infty \left[\int_y^\infty x e^{-x^2/y}\, dx\right] dy$$

$$= \int_0^\infty \left[-\frac{y}{2} e^{-\frac{x^2}{y}}\right]_y^\infty dy = \int_0^\infty \frac{y}{2} e^{-y}\, dy$$

$$= \left[\frac{y}{2}(-e^{-y}) - \frac{1}{2}(e^{-y})\right]_0^\infty = \left[(0-0)-\left(0-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2} \text{ Ans.}$$

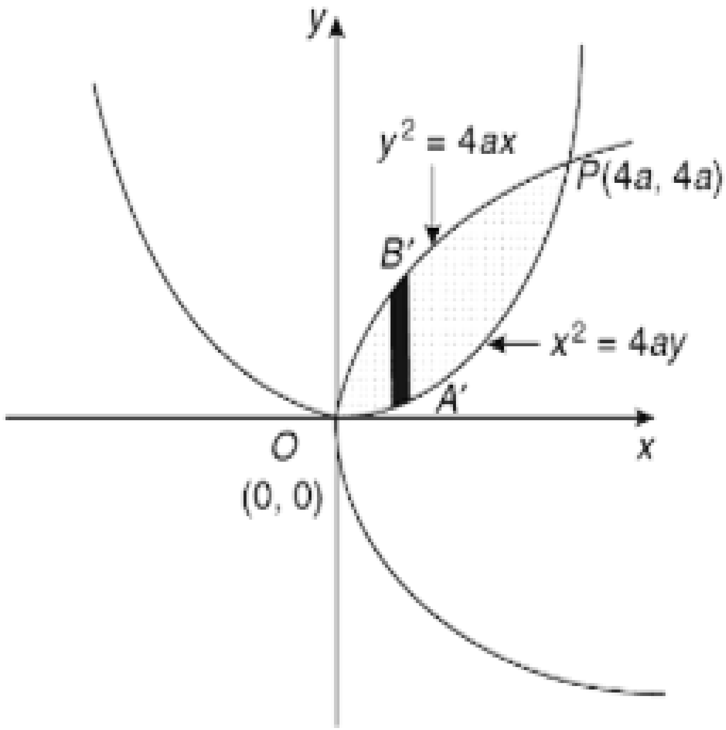
Exp *Change the order of integration and evaluate* $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx.$

Solution

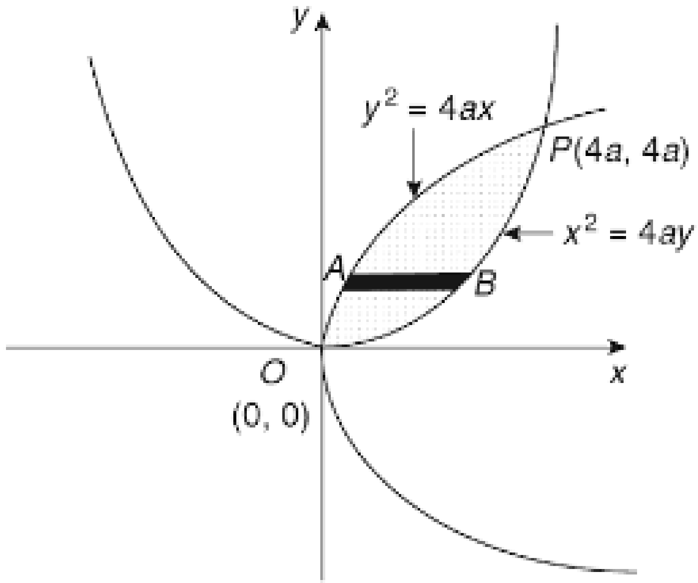
1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y : y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$, along vertical strip $A'B'$
3. The region is bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.
4. The points of intersection of $x^2 = 4ay$ and $y^2 = 4ax$ are obtained as

$$x^4 = 16a^2y^2$$
$$= 16a^2(4ax)$$
$$x(x^3 - 64a^3) = 0$$
$$x = 0, x = 4a$$
$$\therefore y = 0, y = 4a$$

The points of intersection are O (0, 0) and P (4a, 4a).



5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = 4ax$ and terminates on the parabola $x^2 = 4ay$.



Limits of $x : x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

Limits of $y : y = 0$ to $y = 4a$

Hence, the given integral after change of order is

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a}\left|\frac{x^2}{2}\right|_{\frac{y^2}{4a}}^{2\sqrt{ay}} y\,dy$$

$$= \frac{1}{2}\int_0^{4a}\left(4ay - \frac{y^4}{16a^2}\right)dy$$

$$= \frac{1}{2}\left|4a\cdot\frac{y^2}{2} - \frac{1}{16a^2}\cdot\frac{y^5}{5}\right|_0^{4a}$$

$$= \frac{1}{2}\left[2a(16a^2) - \frac{1}{16a^2}\cdot\frac{(4a)^5}{5}\right]$$

$$= \frac{1}{2}\cdot\frac{96}{5}a^3$$

$$= \frac{48}{5}a^3$$

Practice questions

Ques 1: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(\sqrt{1-x^2-y^2})(e^y+1)} \, dx\, dy$ [2011-12]

Ques 2: changing the order of integration in
$I = \int_0^{\infty} \int_{x/4}^{2} f(x,y)\, dy\, dx$, leads to [2016-17]
$I = \int_r^s \int_p^q f(x,y)\, dx\, dy$, say, what is q?

Ques 3: change the order of integration & evaluate $\int_0^2 \int_{x^2/4}^{3-x} xy \, dy\, dx$ [2018-19]

L-30

Area by Double Integral

Introduction: Area by double integration is given by

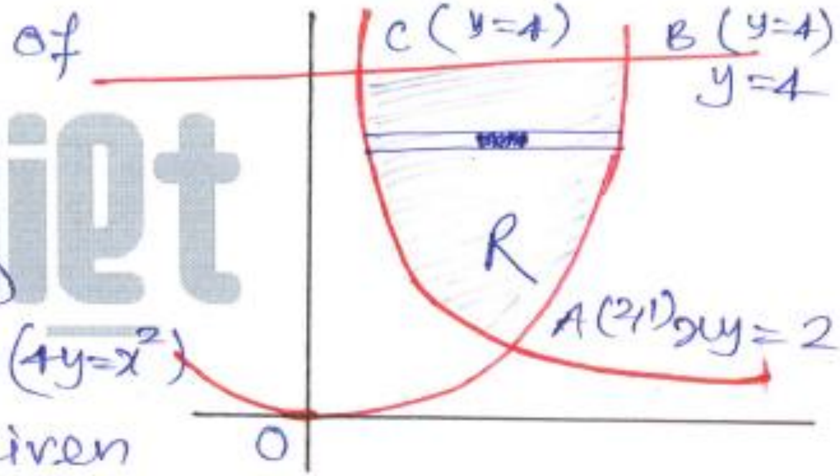
a) Area of region R $= \iint_R dx\,dy$ [in Cartesian]

b) Area of region R $= \iint_R r\,dr\,d\theta$ [in Polar]

Exp: Determine the area bounded by the curves $xy = 2$, $4y = x^2$ & $y = 4$ in 1^{st} quadrant. [2014-15]

Sol^n: Required area of shaded region is

$$A = \iint_R dx\,dy \quad \text{--- (1)}$$



where R is given by $\Rightarrow$ $1 \le y \le 4$, $\frac{2}{y} \le x \le 2\sqrt{y}$

$$A = \int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\,dy = \int_1^4 \left[2\sqrt{y} - \frac{2}{y}\right] dy$$

$$= 2\left[\frac{2}{3} y^{3/2} - \log y\right]_1^4$$

$$= 2\left[\frac{16}{3} - 2\log 2 - \frac{2}{3}\right]$$

$$= \frac{28}{3} - 4\log 2 \quad \text{(Ans).}$$

Exp: Compute the area of lemniscate [2013-14]

$r^2 = a^2 \cos 2\theta$

Soln: Given curve is

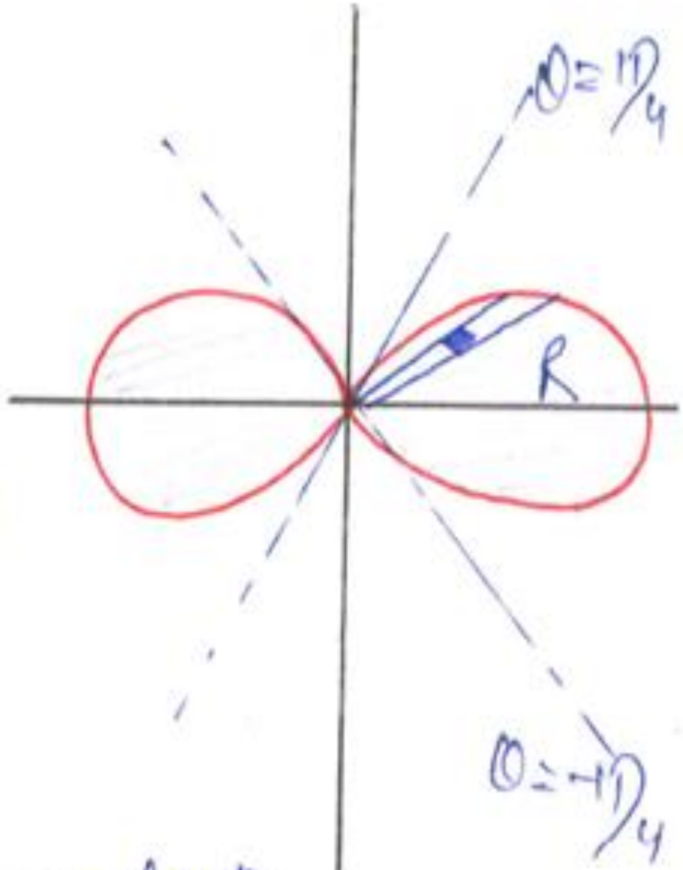
$r^2 = a^2 \cos 2\theta$

or $r = \pm a\sqrt{\cos 2\theta}$

here θ lies between

$\theta = 0$ to $\theta = \pi/4$

Required area



$A = 4 \times$ Area of R in one loop

$= 4 \iint_R r\,dr\,d\theta$

$$A = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta.$$

$$= 4\int_{0}^{\pi/4}\left(\frac{r^2}{2}\right)_{0}^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 4\times\frac{1}{2}\int_{0}^{\pi/4}\left(a\sqrt{\cos 2\theta}\right)^2 d\theta.$$

$$= 2a^2\int_{0}^{\pi/4}\cos 2\theta\,d\theta$$

$$= 2a^2\left(\frac{\sin 2\theta}{2}\right)_{0}^{\pi/4}$$

$$= \frac{2a^2}{2}\left[\sin \pi/2 - 0\right]$$

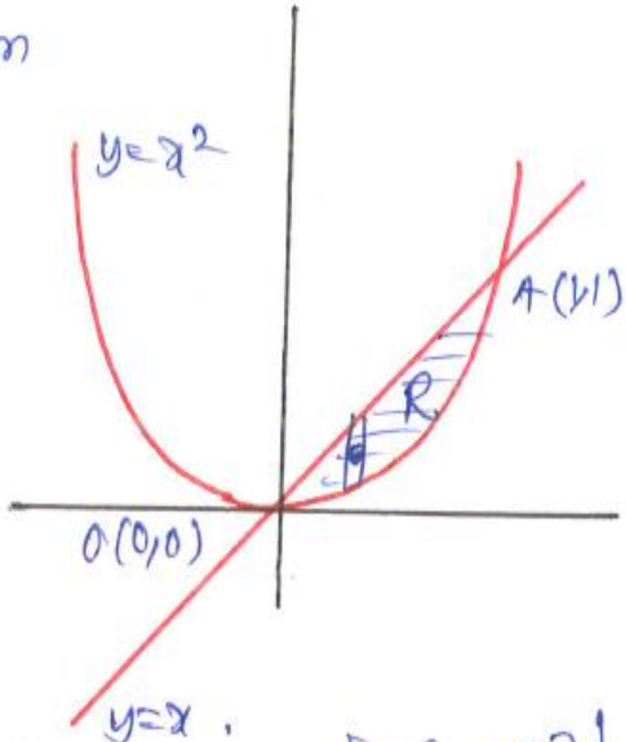
$$= a^2 \qquad \text{Ans.}$$

Exp: Evaluate the area enclosed between the parabola $y=x^2$ & the straight line $y=x$.

Sol^n: Area of the shaded region is

$$A = \iint_R dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} dy\,dx$$



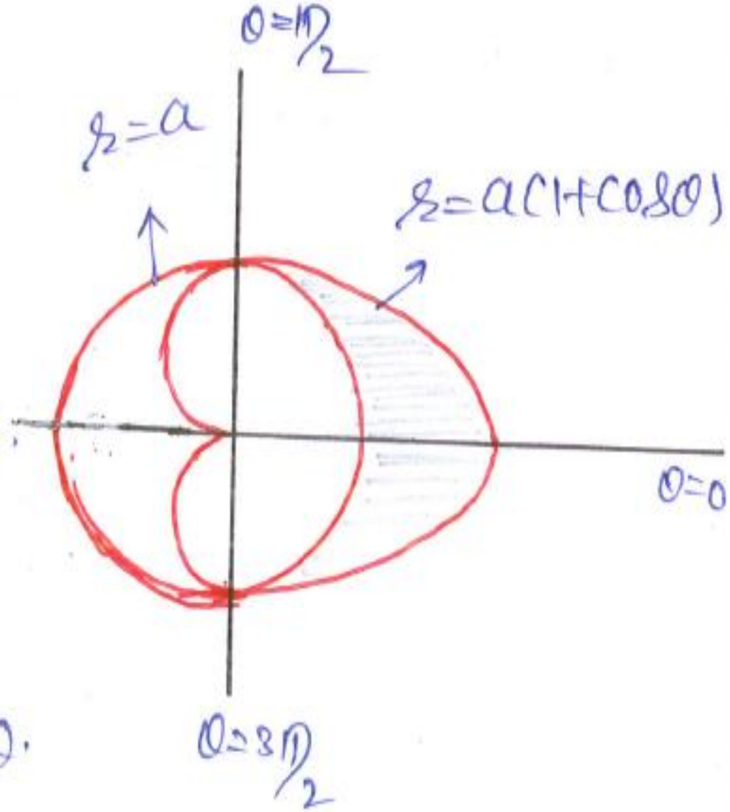
$$= \int_0^1 \left[y \right]_{x^2}^{x} dx$$

$$= \int_0^1 [x-x^2]\,dx = \int_0^1 [x-x^2]\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \text{Ans.}$$

Exp: find the area outside the circle $r=a$ & inside the cardioid $r=a[1+\cos\theta]$

Soln: Area of the shaded region is



$$A = \iint_R r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}\left[a^2(1+\cos\theta)^2 - a^2\right] d\theta.$$

$$= a^2 \int_0^{\pi/2} \left[(1+2\cos\theta + \cos^2\theta - 1) \right] d\theta.$$

$$\text{Area} = a^2 \int_0^{\pi/2} \left[2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta.$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left[1 + 4\cos\theta + \cos 2\theta\right] d\theta.$$

$$= \frac{a^2}{2} \left[\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{2} + 4\right] = \frac{a^2}{4}(\pi + 8) \quad \text{Ans.}$$

Ques :

Example 1. *Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*

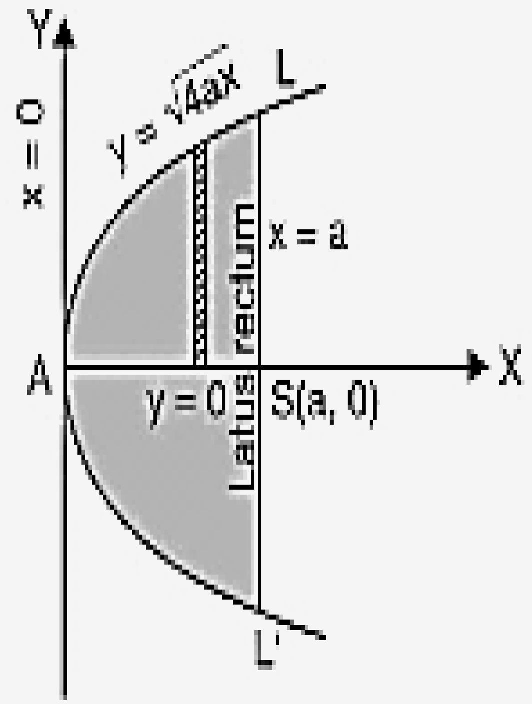
Solution. Required area = 2 (area (ASL)

$$= 2\int_0^a \int_0^{2\sqrt{ax}} dy\, dx$$

$$= 2\int_0^a 2\sqrt{ax}\, dx$$

$$= 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^a = \frac{8a^2}{3}$$

Ans.



HOME WORK QUESTIONS

1) Determine the area of the region bounded by the curves $xy = 2$ $4y = x^2$ $y = 4$.

(Uptu-2001,2008) **Ans=** $\frac{28}{3} - 4\log 2$

2) Find by double integration the area bounded by the pair of axis $y = 2 - x$ and $y^2 = 2(2 - x)$. **Ans=** $2/3$

Question 3. Determine the area of region bounded by the curves

$$xy = 2,\ 4y = x^2,\ y = 4.$$

Answer $\frac{28}{3} - 4\log 2.$

Question 4: Find the area inside the cardioid *r = a(1+ cosθ)* and outside the circle *r = 2acosθ*.

Ans: $\Pi \frac{a^2}{2}$

L-31

Introduction to Triple integration, volume by triple integral

Introduction to Triple integration

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Introduction: Triple integral of a function of three variables $f(x,y,z)$ in a region V is denoted by $I = \iiint_V f(x,y,z)\, dx\, dy\, dz$

Evaluation of Triple integral: If region V is defined by $a \le x \le b$, $c \le y \le d$, $g \le z \le h$ where a, b, c, d, g, h all are constants.

then $I = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=g}^{h} f(x, y, z)\, dz\, dy\, dx$

can be calculated first w.r.t z between g and h, then w.r.t y between c & d & then w.r.t x between a & b.

If V is given by

$a \le x \le b$, $\phi_1(x) \le y \le \phi_2(x)$ &

$f_1(x,y) \le z \le f_2(x,y)$ then

$$I = \int_{x=a}^{b} \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

If z is the inner most integral then we solve the above integral first wrt z by keeping x, y as constant then wrt y by keeping x as a constant & at last wrt x between a to b.

Exp: Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$, where

R: $0 \le x \le 1$, $1 \le y \le 2$; $2 \le z \le 3$ [2015-16, 2017-18]

Sol^n: $I = \int_{x=0}^{1} \int_{y=1}^{2} \int_{z=2}^{3} (x+y+z)\,dz\,dy\,dx$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \left[(x+y)z + \frac{z^2}{2} \right]_2^3 dy\,dx$$

$$= \int_{0}^{1} \int_{y=0}^{2} \left[(x+y) + \frac{5}{2} \right] dy\,dx$$

$$= \int_{0}^{1} \left[\left(x+\frac{5}{2}\right) y + \frac{y^2}{2} \right]_1^2 dx = \int_0^1 \left[\left(x+\frac{5}{2}\right) + \frac{3}{2} \right] dx$$

$$= \left(\frac{x^2}{2} + 4x \right)_0^1 = \frac{9}{2} \quad \text{Ans.}$$

Example 2 *Evaluate* $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz.$

Solution. $I = \int_0^{\log 2}\int_0^{x} e^{x+y}\left[e^{z}\right]_0^{x+y} dx\,dy$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}(e^{x+y}-1)\,dx\,dy = \int_0^{\log 2}\int_0^{x}\left[e^{2(x+y)}-e^{(x+y)}\right]dx\,dy$$

$$= \int_0^{\log 2}\left[e^{2x}.\frac{e^{2y}}{2}-e^{x}.e^{y}\right]_0^{x} dx = \int_0^{\log 2}\left(\frac{e^{4x}}{2}-e^{2x}-\frac{e^{2x}}{2}+e^{x}\right)dx$$

$$= \left[\frac{e^{4x}}{8}-\frac{e^{2x}}{2}-\frac{e^{2x}}{4}+e^{x}\right]_0^{\log 2} = \left[\frac{e^{4\log 2}}{8}-\frac{e^{2\log 2}}{2}-\frac{e^{2\log 2}}{4}+e^{\log 2}\right]-\left(\frac{1}{8}-\frac{1}{2}-\frac{1}{4}+1\right)$$

$$= \left(\frac{e^{\log 16}}{8}-\frac{e^{\log 4}}{2}-\frac{e^{\log 4}}{4}+e^{\log 2}\right)-\left(\frac{1}{8}-\frac{1}{2}-\frac{1}{4}+1\right)$$

$$= \left(\frac{16}{8}-\frac{4}{2}-\frac{4}{4}+2\right)-\left(\frac{1}{8}-\frac{1}{2}-\frac{1}{4}+1\right)=\frac{5}{8}$$

Ans.

Volume by Triple integration

Volume of the solid 'R' is given by

$$\text{Volume} = \iiint_R dx\,dy\,dz$$

Exp: Find the volume of the solid bounded by the surfaces $x=0$, $y=0$, $z=0$ & $x+y+z=1$

[2018-19, 2020-21]

Solⁿ: Volume $= \iiint_R dx\,dy\,dz$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} (z)_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1\int_{y=0}^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 \left[(-x+1)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x)^2\right] dx$$

$$= \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{1}{6} \quad \text{Ans.}$$

Exp: Find the volume of the region bounded by the surfaces $y = x^2$, $x = y^2$ & the planes $z = 0$, $z = 3$ [2019-20]

Sol^n: Volume $= \iiint_R dx\,dy\,dz$ — ①

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{3} dz\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} [z]_0^3\,dy\,dx = 3\int_0^1 (y)_{x^2}^{\sqrt{x}}\,dx$$

$$= 3\int_0^1 [\sqrt{x} - x^2]\,dx$$

$$= 3\int_0^1 \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)_0^1$$

$$= 3\left(\frac{2}{3} - \frac{1}{3}\right)$$

$$= 3\left[\frac{1}{3}\right]$$

$= 1$ cubic unit

Example A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the plane $z = 0$ and surface $z = c + xy$.

Sol. Here x varies from 0 to a

y varies from 0 to $\dfrac{bx}{a}$

z varies from 0 to $c + xy$

Hence, the volume is

$$V = \int_0^a \int_0^{bx/a} \int_0^{c+xy} dx\,dy\,dz = \int_0^a \int_0^{bx/a} (c+xy)\,dx\,dy$$

$$= \int_0^a \left[cy + \frac{xy^2}{2} \right]_0^{bx/a} dx = \int_0^a \left(\frac{bcx}{a} + \frac{b^2x^3}{2a^2} \right) dx$$

$$= \left[\frac{bcx^2}{2a} + \frac{b^2x^4}{8a^2} \right]_0^a = \frac{bca^2}{2a} + \frac{b^2a^4}{8a^2} = \frac{ab}{8}(4c + ab).$$

Practice questions

Ques: Evaluate the triple integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz$$

[2016-17]

Ques: Evaluate $\iiint x^2yz \, dx\, dy\, dz$ throughout the volume bounded by the planes $x=0,\ y=0,\ z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

[2016-17]

Ques: Find the volume of the solid which is bounded by the surfaces $az = x^2+y^2$ & $z=x$ [2011-12]

L-32

Change of variable in double and triple integral

Change of variable in double integral

Let the double integral

$I = \iint_R f(x,y)\,dx\,dy$ —① if it is to be changed in the new variables u & v.

Relation between u, v, x & y is given by

$x = \phi(u,v)$, $y = \psi(u,v)$

Then

$$I = \iint_R f(x,y)\,dx\,dy$$

$$= \iint_{R'} f[\phi(u,v), \psi(u,v)]\,|J|\,du\,dv \quad —②$$

where $dx\,dy = |J|\,du\,dv$ & $J = \frac{\partial(x,y)}{\partial(u,v)}$

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Change of variables from (x, y) to Polar coordinates (r, θ) ⇒

Here $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\,dx\,dy = \iint f[r\cos\theta, r\sin\theta]\, r\,dr\,d\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r \quad \text{so } dx\,dy = r\,dr\,d\theta.$$

Exp: Evaluate $\iint_R (x+y)^2 \, dx\, dy$, where R is the region bounded by the parallelogram in the xy-plane with vertices (1,0), (3,1), (2,2), (0,1), using the transformation

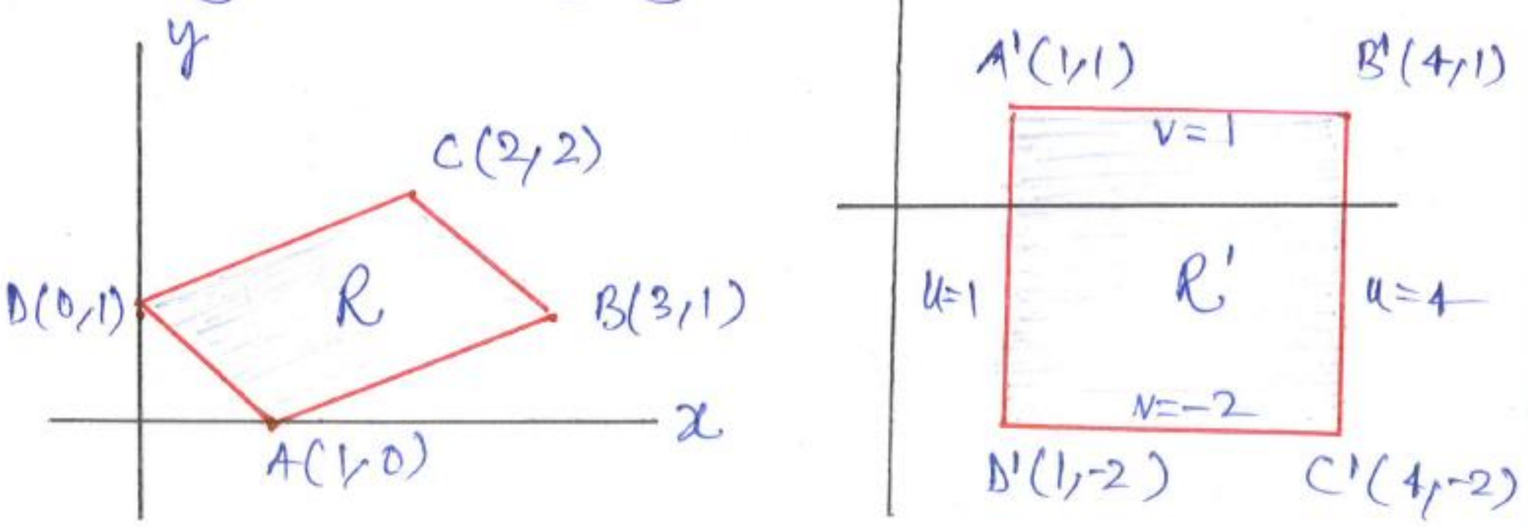
$u = x+y$, $v = x-2y$ [2019-20]

Sol^n: The vertices A(1,0), B(3,1), C(2,2), D(0,1) of the parallelogram ABCD in xy-plane become A'(1,1), B'(4,1), C'(4,−2) & D'(1,−2) in the uv-plane by using the transformations

$u = x+y$ & $v = x-2y$

The region R in xy plane becomes the region R' in the uv-plane which is a rectangle bounded by the line $u=1$, $u=4$, $v=-2$ & $v=1$. Solving the given equations for x & y we get

$$x = \frac{1}{3}(2u+v), \quad y = \frac{1}{3}(u-v)$$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$dx\,dy = |J|\,du\,dv = \left|-\frac{1}{3}\right| du\,dv = \frac{1}{3}\,du\,dv$$

$$\iint_R (x+y)^2\,dx\,dy = \iint_{R'} u^2\,|J|\,du\,dv$$

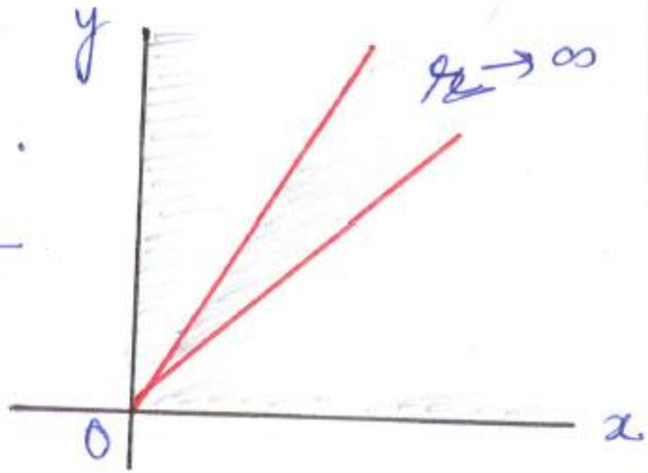
$$= \int_{-2}^{1}\int_{1}^{4} \frac{u^2}{3}\,du\,dv = \int_{-2}^{1} \frac{1}{3}\left(\frac{u^3}{3}\right)_1^4 dv$$

$$= \int_{-2}^{1} 7\,dv = 7 \times 3 = 21 \quad \text{Ans.}$$

Exp: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates, hence show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$ [2018-19]

Sol^n: Given region of integration is I^st quadrant.



To change it into polar we have

$x = r\cos\theta,\ y = r\sin\theta.$

$dx\,dy = r\,dr\,d\theta.$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta.$$

$$= \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta.$$

$$= \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-t}}{2} \, dt \, d\theta \quad , \text{where} \quad t = r^2$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-t} \right]_0^{\infty} d\theta.$$

$$= -\frac{1}{2} \int_0^{\pi/2} (0-1) \, d\theta.$$

$$= \frac{1}{2} (\theta)_0^{\pi/2}$$

$$= \pi/4$$

Now let $I = \int_0^\infty e^{-x^2}\,dx$

between the same limits, we have

$I = \int_0^\infty e^{-y^2}\,dy$

$I^2 = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2}\,dx\,dy$

$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \pi/4$

$I = \frac{\sqrt{\pi}}{2}$

Ques 3. *Using the transformation $x + y = u$, $y = uv$; show that*

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx = \frac{1}{2}(e-1)$$

Sol. Since $x = u(1-v), y = uv$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\therefore \quad dx\, dy = |J|\, du\, dv = u\, du\, dv$$

Also,

$x = 0 \Rightarrow u(1-v) = 0 \Rightarrow u = 0, v = 1$

$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0, v = 0$

$x + y = 1 \Rightarrow u = 1$

Hence the limits of u are 0 to 1 and the limits of v are 0 to 1.

$$\therefore \int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx = \int_0^1 \int_0^1 e^{uv/u} |J|\, du\, dv$$

$$= \int_0^1 \int_0^1 u e^v\, du\, dv = \left(\frac{u^2}{2}\right)_0^1 \left(e^v\right)_0^1 = \frac{1}{2}(e-1).$$

Example 4. *Evaluate* $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$

Solution. $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$

Limits of $y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0$ ...(1)

(1) represents a circle whose centre is (1, 0) and radius = 1.

Lower limit of y is 0 *i.e.*, x-axis.

Region of integration is upper half circle.

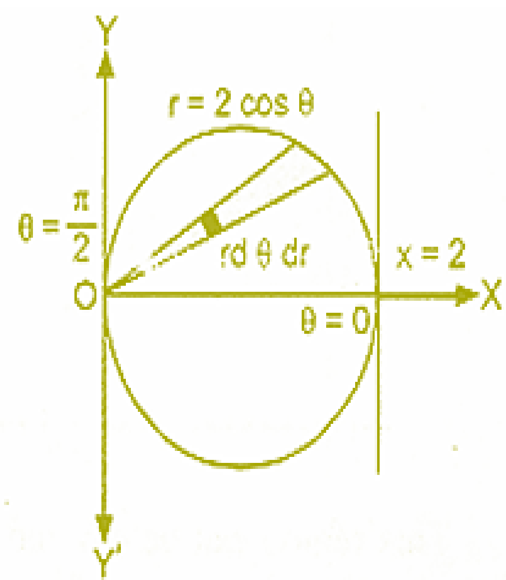
Let us convert (1) into polar co-ordinates by putting

$x = r\cos\theta,\ y = r\sin\theta$

$$r^2 - 2r\cos\theta = 0 \Rightarrow r = 2\cos\theta$$

Limits of r are 0 to $2\cos\theta$

Limits of θ are 0 to $\frac{\pi}{2}$



$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 (r\, d\theta\, dr)$$

$$= \int_0^{\pi/2} d\theta \int_0^{2\cos\theta} r^3\, dr$$

$$= \int_0^{\pi/2} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$$

$$= 4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4 \times \frac{3 \times 1 \times \pi}{4 \times 2 \times 2}$$

$$= \frac{3\pi}{4} \text{ Ans.}$$

Exp: Evaluate the following by changing into polar coordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$

Soln: changing to polar coordinates

we have $x = r\cos\theta$, $y = r\sin\theta$

so $x^2+y^2 = r^2 \Rightarrow r^2 = a^2$

$$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_0^{\pi/2} \int_0^a r^2 \sin^2\theta \cdot r \cdot r\, dr\, d\theta.$$

$$= \int_0^{\pi/2} \sin^2\theta \cdot \left(\frac{r^5}{5}\right)_0^a d\theta.$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta.$$

$$= \frac{a^5}{10} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}$$

Change of variables in Triple integral

$I = \iiint_V f(x,y,z)\, dx\, dy\, dz$, can be changed into the variables u, v, w as

$$I = \iiint_{V'} f\left[\phi_1(x,y,z), \phi_2(x,y,z), \phi_3(x,y,z)\right] |J|\, du\, dv\, dw$$

where $u = \phi_1(x,y,z)$, $v = \phi_2(x,y,z)$

$w = \phi_3(x,y,z)$

& $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Change of cartesian coordinates (x, y, z) to spherical polar coordinate:

If we have $I = \iiint_R f(x,y,z)\, dx\,dy\,dz$ —(1)

Then to change above integral in spherical polar coordinates, we have

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

so $dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\phi$.

so (1) will be

$$I = \iiint_{R'} F(r,\theta,\phi)\, r^2\sin\theta\, dr\,d\theta\,d\phi \quad -(2)$$

Spherical Polar coordinate system:

$x = r \sin\theta \cos\phi$

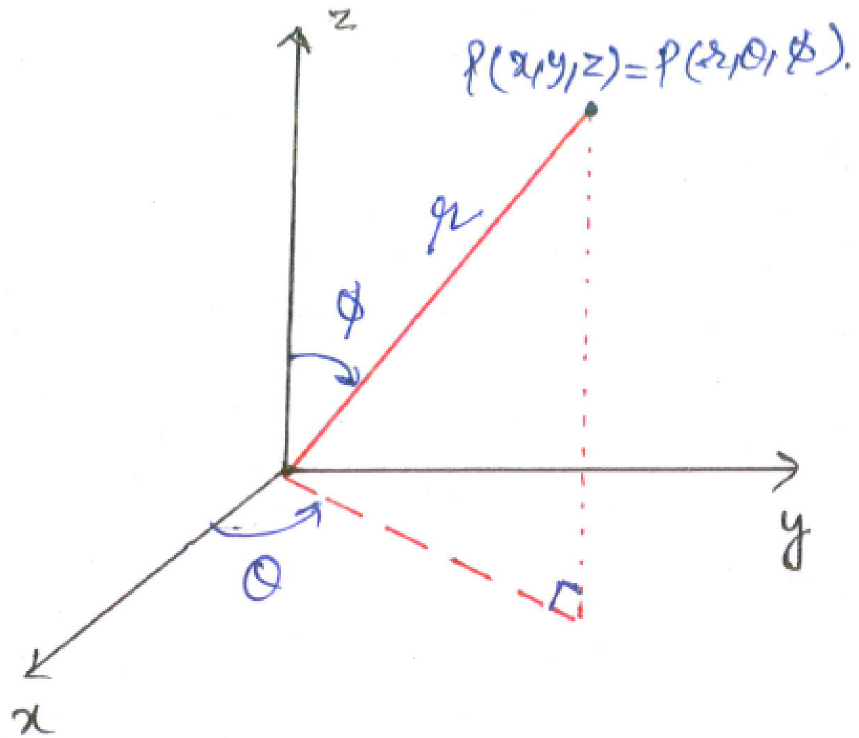
$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

general equation of a sphere is:

$$x^2 + y^2 + z^2 = r^2$$

at origin:



Ques: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar coordinates.

[2015-16]

Sol^n: here region of integration is bounded by $0 \le z \le \sqrt{1-x^2-y^2}$,

$0 \le y \le \sqrt{1-x^2}$ & $0 \le x \le 1$

so ($x^2+y^2+z^2 = r^2$, sphere of radius 1)

region is in Ist octant only

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left[\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right] \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[\sin^{-1} r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1} r \right) \right]_0^1 d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left(\frac{\pi}{2} - \frac{\pi}{4} \right) d\theta \, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \left(-\cos\theta \right)_0^{\pi/2} d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} d\phi$$

$$= \frac{\pi^2}{8}$$

Example- *Evaluate* $\iiint \frac{z^2 dx\, dy\, dz}{x^2+y^2+z^2}$ *over the volume of the sphere* $x^2+y^2+z^2=2$.

Solution- Here, we have $I = \iiint \frac{z^2 dx\, dy\, dz}{x^2+y^2+z^2}$ ...(1)

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $dx\, dy\, dz = r^2\sin\theta\, dr\, d\theta\, d\phi$ in (1), we get

[The limits r, θ and ϕ over the first octant of $x^2+y^2+z^2=r^2$ are $0, \sqrt{2}$; $0, \frac{\pi}{2}$ and $0, \frac{\pi}{2}$].

$$I = 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} \frac{r^4\cos^2\theta\sin\theta}{r^2}\, dr\, d\theta\, d\phi$$

$$I = 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} \frac{r^4\cos^2\theta\sin\theta}{r^2}\,dr\,d\theta\,d\phi$$

$$= 8\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \cos^2\theta\sin\theta\,d\theta \,.\, \int_0^{\sqrt{2}} r^2 dr$$

$$= 8\,[\phi]_0^{\pi/2}\left[-\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} = 8\,\frac{\pi}{2}\cdot\frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{8\pi\sqrt{2}}{9}.\quad \text{Ans.}$$

Example

Evaluate the integral $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ *taken over the volume enclosed by the sphere* $x^2+y^2+z^2=1$.

Solution. Let us convert the given integral into spherical polar co-ordinates. By putting

$$x = r\sin\theta\cos\phi\,;\quad y = r\sin\theta\sin\phi\,;\quad z = r\cos\theta$$

$$\iiint (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \int_0^1 r^4 dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \left(\frac{r^5}{5}\right)_0^1 = \frac{1}{5}\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi} = \frac{2}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2}{5}(\phi)_0^{2\pi} = \frac{4\pi}{5}$$

Ans.

Practice questions

Ques: Evaluate by changing the variables $\iint (x+y)^2 \, dx\,dy$, where R is the region bounded by the lines $x+y=0$, $x+y=2$, $3x-2y=0$, $3x-2y=3$

[2013-14], [2020-21]

Ques: Evaluate $\iint (x-y)^4 \exp(x+y) \, dx\,dy$, where R is the square in the x-y plane with vertices at (1,0), (2,1), (1,2) & (0,1).

[2012-13]

Practice question

Ques: If the volume of an object expressed in the spherical coordinates as following:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} r^2 \sin\phi \, dr \, d\phi \, d\theta$$

. Evaluate the value of V. [2016-17]

Unit IV
(Multivariate Calculus-I)
L-33

GROUP OF INSTITUTIONS

Gamma Function and Beta Function

Gamma Function

If n is positive, then the definite integral $\int_0^\infty e^{-x} x^{n-1}\, dx$, which is a function of n, is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx,\ n > 0.$$

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left| -e^{-x} \right|_0^\infty = 1.$

Some Basic Formulae

1. $\Gamma(n+1) = n\Gamma(n)$

2. $\Gamma(n+1) = n!$ **when n is a positive integer**

3. If n is a positive fraction, then by repeated application of above formula, we get

$$\Gamma(n) = (n-1)(n-2) \times \text{go on decreasing by } 1 \ldots\ldots$$

the series of factors being continued so long as the factors remain positive, multiplied by Γ (last factor).

$$\Gamma\left(\frac{11}{4}\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

4. $\left|\frac{1}{2}\right. = \sqrt{\pi}$

TRANSFORMATION OF GAMMA FUNCTION

MIET GROUP OF INSTITUTIONS

We know that $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$ ...(1)

(*i*) Replace x by $k\,y$, so that $dx = k\,dy$; then (1) becomes

$$\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$$

(ii) Putting $e^{-x} = y$, so that $-e^{-x} dx = dy$ and $-x = \log y$, $x = \log \frac{1}{y}$, (1) becomes

$$\Gamma n = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y \cdot \frac{dy}{e^{-x}}$$

$$= \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} y \cdot \frac{dy}{y}$$

$$= \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

Example . *Evaluate* $\Gamma\left(-\frac{1}{2}\right)$.

Solution. $\Gamma(n+1) = n\,\Gamma(n)$

$$\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\,\Gamma\left(-\frac{1}{2}\right) \Rightarrow \Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\,\Gamma\left(-\frac{1}{2}\right) \Rightarrow \sqrt{\pi} = -\frac{1}{2}\,\Gamma\left(-\frac{1}{2}\right) \Rightarrow \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$ **Ans.**

Beta Function

If m, n are positive, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, which is a function of m and n, is called the Beta Function (or Eulerian integral of first kind) and is denoted by $\beta(m, n)$. Thus,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m > 0, n > 0.$$

Note

- Beta function is a symmetric function. i.e. $B(m, n) = B(n, m)$, where $m > 0, n > 0$

Transformations of Beta Function

(1) $$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ | Put $x = \frac{1}{1+y}$ $\therefore$ $dx = -\frac{1}{(1+y)^2}\, dy$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1 - \frac{1}{1+y}\right)^{n-1}\left\{\frac{-1}{(1+y)^2}\right\} dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2}\, dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$\therefore$ $$\beta(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

But $\beta(m, n) = \beta(n, m)$

$\therefore$ $$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

(2) $$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Proof. $$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \sin^2\theta$

$\therefore \quad dx = 2\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

SYMMETRY OF BETA FUNCTION i.e., $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \; m > 0, n > 0$$

Since $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore \quad \beta(m, n) = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = \beta(n, m)$$

Hence, $\boxed{\beta(m, n) = \beta(n, m).}$

RELATION BETWEEN BETA *AND* GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ [A.K.T.U. 2018, 2019, 2022]

Method-I to Prove Relation

$$\Gamma(n) = k^n \int_0^\infty e^{-kx} x^{n-1}\, dx$$

$$= z^n \int_0^\infty e^{-zx} x^{n-1}\, dx$$ (Replace k by z)

Multiplying both sides by $e^{-z} z^{m-1}$, we get

$$\Gamma(n) \cdot e^{-z} z^{m-1} = \int_0^\infty z^n \cdot e^{-zx} \cdot x^{n-1} \cdot e^{-z} \cdot z^{m-1}\, dx = \int_0^\infty z^{n+m-1} e^{-z(1+x)} x^{n-1}\, dx$$

Integrating both sides w.r.t. z from 0 to ∞, we get

$$\Gamma(n)\int_0^\infty e^{-z} z^{m-1}\,dz = \int_0^\infty x^{n-1}\left\{\int_0^\infty e^{-z(1+x)}\, z^{m+n-1}\,dz\right\} dx$$

$$\Rightarrow \quad \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left\{\int_0^\infty e^{-y}\cdot\frac{y^{m+n-1}}{(1+x)^{m+n-1}}\,\frac{dy}{(1+x)}\right\} dx$$

where $z(1+x) = y$ so that $dz = \dfrac{dy}{1+x}$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left\{\int_0^\infty e^{-y}\, y^{m+n-1}\,dy\right\} dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx\;\Gamma(m+n) = \Gamma(m+n)\;\beta(m,n)$$

$$\therefore \quad \boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Method-II to Prove Relation

We know that $\Gamma(m) = \int_0^{\infty} e^{-t}\, t^{m-1}\, dt$

Putting $t = x^2$ so that $dt = 2x\, dx$

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx \qquad \ldots(1)$$

Similarly, $\Gamma(n) = 2\int_0^{\infty} e^{-y^2}\, y^{2n-1}\, dy$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx \cdot \int_0^{\infty} e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1}\, y^{2n-1}\, dx\, dy$$

Changing to polar co-ordinates, we have

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \, dr\, d\theta$$

$$= 4\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1}\, dr \cdot \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \qquad ...(2)$$

$$= \left[2\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1}\, dr\right] \cdot \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta\right]$$

| Using (2) of 4.18

$$= \Gamma(m+n)\,\beta(m, n)$$

Hence, $$\boxed{\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Some Important Results

(i)

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

(ii) If $\int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$, where $0 < n < 1$,

then

$$\Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Example *Prove that* $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Solution. Putting $n = \frac{1}{4}$ in result of example 22, we obtain

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\Rightarrow \qquad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$\Rightarrow \quad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$$ **Proved.**

DUPLICATION FORMULA

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{(2)^{2m-1}}\Gamma(2m)$$ **where m is positive.** (*M.T.U. 2013;*

Proof. We have

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \quad (1)$$

Putting $2n-1=0$ or $n=\frac{1}{2}$ in (1), we obtain

$$\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta=\frac{\Gamma(m)\sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)} \quad (2)$$

Again putting $n=m$ in equation (1), we obtain

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2}(2\sin\theta\cos\theta)^{2m-1}\,d\theta=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

$$\frac{1}{2^{2m}}\int_0^{\pi/2}(\sin 2\theta)^{2m-1}\cdot 2d\theta=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

Putting $2\theta=\phi$ so that $2\,d\theta=d\phi$, this reduces to

$$\frac{1}{2^{2m}}\int_0^{\pi}\sin^{2m-1}\phi\,d\phi=\frac{(\Gamma\,m)^2}{2\,\Gamma\,(2m)}$$

$$\frac{2}{2^{2m}}\int_0^{\pi/2}\sin^{2m-1}\phi\,d\phi=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

Replacing ϕ by θ, we finally obtain

$$\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta=\frac{2^{2m-1}\,(\Gamma m)^2}{2\,\Gamma(2m)} \tag{3}$$

From (2) and (3), we get

$$\frac{\Gamma(m)\sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)}=\frac{2^{2m-1}\,(\Gamma m)^2}{2\,\Gamma(2m)}$$

$$\Rightarrow \qquad \boxed{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}$$

Home Work

Prove the following problems:

1 $$\frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})} = (2)^{1/3}\sqrt{\pi} \quad (A.K.T.U.\ 2017)$$

2 $$\Gamma(-\tfrac{3}{2}) = \frac{4}{3}\sqrt{\pi}$$

3 $$\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}, (p>0, q>0)$$

4 $$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1), for\ m>0, n>0$$

Unit IV
(Multivariate Calculus-I)
L-34

Problems Based Upon Beta & Gamma Functions

Gamma Function

If n is positive, then the definite integral $\int_0^\infty e^{-x} x^{n-1}\, dx$, which is a function of n, is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx,\ n > 0.$$

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left| -e^{-x} \right|_0^\infty = 1.$

Beta Function

If m, n are positive, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, which is a function of m and n, is called the Beta Function (or Eulerian integral of first kind) and is denoted by $\beta(m, n)$. Thus,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0.$$

Note

- Beta function is a symmetric function. i.e. $B(m,n) = B(n,m)$, where $m > 0, n > 0$

Using Beta and Gamma functions, evaluate:

(i) $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\,dx$

(ii) $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$ (A.K.T.U. 2014, 2018)

(iii) $\int_0^1 x^5 (1-x^3)^{10}\,dx$

(i) Let $$I = \int_0^{\infty} x^{1/4} e^{-\sqrt{x}} \, dx$$

Put $\sqrt{x} = y \Rightarrow x = y^2$ so that $dx = 2y\,dy$ then equation (1) becomes

$$I = \int_0^{\infty} y^{1/2} e^{-y} \cdot 2y\,dy = 2\int_0^{\infty} e^{-y}\, y^{3/2}\, dy$$

$$= 2\int_0^{\infty} e^{-y}\, y^{(5/2)-1} dy = 2\,\Gamma(5/2) \qquad \text{| By definition}$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} = \frac{3}{2}\sqrt{\pi} \qquad \text{| } \because \Gamma(n+1) = n\Gamma(n)$$

(*ii*) Let

$$I = \int_0^1 x^{3/2}(1-x^3)^{-1/2}\, dx \qquad \ldots(1)$$

Put $x^3 = y \Rightarrow x = y^{1/3}$ so that $dx = \frac{1}{3} y^{-2/3}\, dy$ then equation (1) becomes

$$I = \int_0^1 y^{1/2}(1-y)^{-1/2} \cdot \frac{1}{3} y^{-2/3}\, dy = \frac{1}{3}\int_0^1 y^{-1/6}(1-y)^{-1/2}\, dy$$

$$= \frac{1}{3}\int_0^1 y^{\left(\frac{5}{6}\right)-1}(1-y)^{\left(\frac{1}{2}\right)-1}\, dy = \frac{1}{3}\beta\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$= \frac{1}{3}\frac{\Gamma(5/6)\,\Gamma(1/2)}{\Gamma(4/3)} \qquad \left|\because\ \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right.$$

$$= \frac{\sqrt{\pi}}{3} \cdot \frac{\Gamma(5/6)}{\frac{1}{3}\Gamma(1/3)} = \sqrt{\pi} \cdot \frac{\Gamma(5/6)\,\Gamma(1/6)\,\Gamma(2/3)}{\Gamma(1/6)\,\Gamma(1/3)\,\Gamma(2/3)}$$

$$= \sqrt{\pi} \cdot \frac{\Gamma(2/3)}{\Gamma(1/6)} \cdot \frac{\pi}{\sin\frac{\pi}{6}} \cdot \frac{\sin\frac{\pi}{3}}{\pi}$$

$$= \sqrt{3\pi}\,\frac{\Gamma(2/3)}{\Gamma(1/6)}$$

(iii) Solution is similar as (ii)

Evaluate:

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx \qquad \text{(M.T.U. 2013)}$$

Sol. (*i*) $$I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9, 15) - \beta(15, 9) = 0$$

Evaluate: $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$ (G.B.T.U. 2011)

Sol. $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

$\therefore \quad dx = \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta \, d\theta$

$$I = \int_0^{\pi/4} \frac{1}{\sec\theta} \cdot \frac{\sec^2\theta}{2\sqrt{\tan\theta}} \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}} = \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Put $2\theta = t \quad \therefore \quad d\theta = \frac{dt}{2}$

$$I = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t\, dt$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{(-1/2)+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{(-1/2)+0+2}{2}\right)} = \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{\Gamma(1/4)^2}{\Gamma(1/4)\,\Gamma(3/4)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{\Gamma(1/4)^2}{\left(\frac{\pi}{\sin \pi/4}\right)} = \frac{1}{8\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2.$$

Example

Prove the following $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi.$ (U.P.T.U. 2014)

Proof: We have $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{3}{4})} \times \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{5}{4})} = \frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{4} \times \frac{\sqrt{\pi}}{\frac{1}{4}\Gamma(\frac{1}{4})} = \pi$$

Example

Prove that.

$$\beta(l, m) \cdot \beta(l + m, n) \cdot \beta(l + m + n, p) = \frac{\Gamma l \, \Gamma m \, \Gamma n \, \Gamma p}{\Gamma(l + m + n + p)}$$

Proof:

$$\text{LHS} = \beta(l, m) \cdot \beta(l + m, n) \cdot \beta(l + m + n, p)$$

$$= \frac{\Gamma l \, \Gamma m}{\Gamma(l + m)} \cdot \frac{\Gamma(l + m) \, . \, \Gamma n}{\Gamma(l + m + n)} \cdot \frac{\Gamma l + m + n \, \Gamma p}{\Gamma(l + m + n + p)}$$

$$= \frac{\Gamma l \, \Gamma m \, \Gamma n \, \Gamma p}{\Gamma(l + m + n + p)} = \text{RHS}$$

Home Work

Show the following:

1 $$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$$

2 $$\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx = \frac{1}{5005}$$

3 $$\int_0^2 (8-x^3)^{-1/3}\, dx = \frac{2\pi}{3\sqrt{3}}$$

4 $$\int_0^1 \frac{x^2\, dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}.$$

5 $$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1),\ for\ m > 0, n > 0$$

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DIRICHLET'S INTEGRAL

DIRICHLET'S INTEGRAL

If V is a region bounded by $x \geq 0,\ y \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(m)}{\Gamma(l+m+n+1)}$$

This integral is known as **Dirichlet's integral** This is an important integral useful in evaluating multiple integrals.

Example . Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Sol. The volume in the positive octant will be

$$V = \iiint dx\, dy\, dz$$

For points within positive octant, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1.$

Put $\frac{x^2}{a^2} = u$ or $x = a\sqrt{u}$, $y = b\sqrt{v}$, $z = c\sqrt{w}$,

$$\therefore \quad dx = \frac{a}{2}u^{-\frac{1}{2}}\, du,\; dy = \frac{b}{2}v^{-\frac{1}{2}}\, dv,\; dz = \frac{c}{2}w^{-\frac{1}{2}}\, dw$$

$$V = \frac{abc}{8}\iiint u^{\left(\frac{1}{2}-1\right)} v^{\left(\frac{1}{2}-1\right)} w^{\left(\frac{1}{2}-1\right)} du\, dv\, dw\text{, where } u+v+w \le 1$$

$$= \frac{abc}{8}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \frac{abc}{8}\,\frac{\left(\sqrt{\pi}\right)^3}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}} = \frac{\pi\, abc}{6}$$

$$\therefore \quad \text{Total volume} = 8 \times \frac{\pi\, abc}{6} = \frac{4}{3}\pi\, abc.$$

Example 2. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axis in A, B and C. Apply Dritchlet's integral to find the volume of the tetrahedran $OABC$. Also find its mass if the density at any point is $kxyz$.

Sol. Let $\frac{x}{a}=u,\ \frac{y}{b}=v,\ \frac{z}{c}=w$, then $u \geq 0,\ v \geq 0,\ w \geq 0$ and $u+v+w \leq 1$

Also, $dx = a\,du,\ dy = b\,dv,\ dz = c\,dw.$

$$\text{Volume } OABC = \iiint_D dx\,dy\,dz$$

$$= \iiint_D abc\,du\,dv\,dw, \text{ where } u+v+w<1$$

$$= abc\iiint_D u^{1-1}v^{1-1}w^{1-1}du\,dr\,dw$$

$$= abc\frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)}=\frac{abc}{3!}=\frac{abc}{6}$$

$$\text{Mass} = \iiint_D kxyz\,dx\,dy\,dz = \iiint_{D'} k\,(au)(bv)(cw)\,abc\,du\,dv\,dw$$

$$= ka^2b^2c^2\iiint_{D'} u^{2-1}v^{2-1}w^{2-1}du\,dv\,dw$$

$$= ka^2b^2c^2\frac{\Gamma 2\,\Gamma 2\,\Gamma 2}{\Gamma(2+2+2+1)} = ka^2b^2c^2\frac{1!1!1!}{6!} = \frac{ka^2b^2c^2}{720}.$$

Example 3. *Find the volume of the solid surrounded by the surface*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1. \qquad \text{(U.P.T.U. 2008)}$$

Sol. Let

$$\left(\frac{x}{a}\right)^{2/3} = u \Rightarrow x = au^{3/2} \quad \therefore \quad dx = \frac{3a}{2} u^{1/2}\, du$$

$$\left(\frac{y}{b}\right)^{2/3} = v \Rightarrow y = bv^{3/2} \quad \therefore \quad dy = \frac{3b}{2} v^{1/2}\, dv$$

$$\left(\frac{z}{c}\right)^{2/3} = w \Rightarrow z = cw^{3/2} \quad \therefore \quad dz = \frac{3c}{2} w^{1/2}\, dw$$

For the positive octant,

$$x \geq 0 \Rightarrow au^{3/2} \geq 0 \Rightarrow u \geq 0$$

$$y \geq 0 \Rightarrow bv^{3/2} \geq 0 \Rightarrow v \geq 0$$

$$z \geq 0 \Rightarrow cw^{3/2} \geq 0 \Rightarrow w \geq 0$$

Then, we have $u + v + w = 1$, $u \geq 0$, $v \geq 0$, $w \geq 0$.

Required volume $= 8\iiint dx\, dy\, dz$

$$= 8\iiint \frac{3a}{2} u^{1/2} . \frac{3b}{2} v^{1/2} . \frac{3c}{2} w^{1/2}\, du\, dv\, dw$$

$$= 27\, abc \iiint u^{\frac{3}{2}-1} v^{\frac{3}{2}-1} w^{\frac{3}{2}-1}\, du\, dv\, dw$$

$$= 27\, abc \,.\, \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} = \frac{4\pi abc}{35}$$

Example **3** . *Find the mass of an octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$, *the density at any point being* $\rho = k\,x\,y\,z$.

Solution. Mass $= \iiint \rho\, dv = \iiint (k\,x\,y\,z)\, dx\, dy\, dz$

$$= k \iiint (xdx)\,(ydy)\,(zdz) \qquad \dots (1)$$

Putting $\frac{x^2}{a^2} = u,\ \frac{y^2}{b^2} = v,\ \frac{z^2}{c^2} = w$ and $u+v+w = 1$

so that $\frac{2xdx}{a^2} = du,\quad \frac{2\,y\,dy}{b^2} = dv,\quad \frac{2\,z\,dz}{c^2} = dw$

$$\text{Mass} = k \iiint \left(\frac{a^2\,du}{2}\right)\left(\frac{b^2\,dv}{2}\right)\left(\frac{c^2\,dw}{2}\right)$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint du\,dv\,dw \quad \text{where} \quad u+v+w \le 1$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

$$= \frac{k\,a^2\,b^2\,c^2}{8}\,\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 3+1} = \frac{k\,a^2\,b^2\,c^2}{8\times 6}$$

$$= \frac{k\,a^2\,b^2\,c^2}{48}$$

Ans.

Example4 *The plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* (A.K.T.U. 2012, 2018)

Sol. Put $\frac{x}{a}=u, \frac{y}{b}=v, \frac{z}{c}=w$, then $u \geq 0, v \geq 0, w \geq 0$ and $u+v+w \leq 1$.

Also, $dx = a\,du, dy = b\,dv, dz = c\,dw$.

$$\text{Volume OABC} = \iiint_D dx\,dy\,dz$$

$$= \iiint_{D'} abc\,du\,dv\,dw, \quad \text{where } u+v+w \leq 1$$

$$= abc \iiint_{D'} u^{1-1} v^{1-1} w^{1-1}\,du\,dv\,dw$$

$$= abc \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3!} = \frac{abc}{6}$$

$$\text{Mass} = \iiint_D kxyz\,dx\,dy\,dz = \iiint_{D'} k(au)(bv)(cw)\,abc\,du\,dv\,dw$$

$$= ka^2b^2c^2 \iiint_{D'} u^{2-1}v^{2-1}w^{2-1}\,du\,dv\,dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} = ka^2b^2c^2 \frac{1!\,1!\,1!}{6\,!} = \frac{ka^2b^2c^2}{720}.$$

Home Work

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1. Evaluate $\iiint xyz \, dx \, dy \, dz$ for all positive value of variables of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

2. Evaluate $\iiint_V (ax^2 + by^2 + cz^2) \, dx \, dy \, dz$ where V is the region bounded by $x^2 + y^2 + z^2 \leq 1$.

3. Compute $\iiint_v x^2 \, dx \, dy \, dz$ over valume of tetraheron bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

4. Evaluate $\iiint_v x^2 yz \, dx \, dy \, dz$ throughout the volume bounded by planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

ANSWERS

1. $$\frac{a^2b^2c^2}{48}$$

2. $$\frac{\pi(a+b+c)}{30}$$

3. $$\frac{a^3bc}{60}$$

4. $$\frac{a^3b^2c^2}{2520}$$

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LIOUVILLE'S EXTENSION OF DIRICHLET'S INTEGRAL

DIRICHLET'S INTEGRAL

If V is a region bounded by $x \geq 0,\ y \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(m)}{\Gamma(l+m+n+1)}$$

This integral is known as **Dirichlet's integral** This is an important integral useful in evaluating multiple integrals.

LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables x, y, z are all positive such that $h_1 < (x+y+z) < h_2$ *then*

$$\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u)\,u^{l+m+n-1}\,du.$$

Example 1. *Evaluate* $\iiint \log(x+y+z)\,dx\,dy\,dz$, *the integral extending over all positive and zero values of x, y, z subject to* $x+y+z<1$.

Sol. $0 \le x+y+z < 1$

$$\therefore \quad \iiint \log(x+y+z)\,dx\,dy\,dz = \iiint x^{1-1}y^{1-1}z^{1-1}\log(x+y+z)\,dx\,dy\,dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(3)}\int_0^1 t^{1+1+1-1}\log t\,dt \qquad \text{| By Liouville's extension}$$

$$= \frac{1}{2}\int_0^1 t^2\log t\,dt = \frac{1}{2}\left[\left(\frac{t^3}{3}\log t\right)_0^1 - \int_0^1 \frac{t^3}{3}\cdot\frac{1}{t}\,dt\right] = \frac{1}{2}\left[-\frac{1}{3}\left(\frac{t^3}{3}\right)_0^1\right] = -\frac{1}{18}.$$

Example . *Show that* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$, *the integral being taken throughout the volume bounded by planes* $x = 0,\ y = 0,\ z = 0$ *and* $x + y + z = 1$.

Sol. $0 \le x + y + z \le 1$

$$\therefore \quad \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint \frac{x^{1-1}y^{1-1}z^{1-1}}{(x+y+z+1)^3}\,dx\,dy\,dz$$

$$= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)}\int_0^1 \frac{1}{(u+1)^3}u^{1+1+1-1}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{u^2}{(u+1)^3}\,du \qquad \text{| By Liouville's extension}$$

Put $u + 1 = t$ so that $du = dt$

$$\therefore \quad \text{Required integral} = \frac{1}{2}\int_1^2 \frac{(t-1)^2}{t^3}\,dt = \frac{1}{2}\int_1^2 \left(\frac{t^2-2t+1}{t^3}\right)dt$$

$$= \frac{1}{2}\int_1^2 \left(\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3}\right)dt = \frac{1}{2}\left[\log t + \frac{2}{t} - \frac{1}{2t^2}\right]_1^2$$

Example *Find the value of* $\iiint \log(x+y+z)\,dx\,dy\,dz$ *the integral extending over all positive and zero values of x, y, z subject to the condition* $x+y+z<1$.

Solution. By Liouville's theorem when $0 < x + y + z < 1$

$$\iiint \log(x+y+z)\,dx\,dy\,dz$$

$$= \iiint \log(x+y+z)\,x^{1-1}y^{1-1}z^{1-1}\,dx\,dy\,dz = \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1)}\int_0^1 (\log u)u^{1+1+1-1}\,du$$

$$= \frac{1}{\Gamma 3}\int_0^1 u^2\log u\,du = \frac{1}{2}\left[\log u\left(\frac{u^3}{3}\right) - \frac{1}{3}\frac{u^3}{3}\right]_0^1 = \frac{1}{2}\left(-\frac{1}{9}\right) = -\frac{1}{18}$$

Ans.

Home work

1. Find the mass of a solid $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$, the density at any point being $\rho = kx^{l-1} y^{m-1} z^{n-1}$ where x, y, z are all positive.

2. Find the mass of the region bounded by ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if the density varies as the square of the distance from the centre. **[Hint.** $\rho = k(x^2 + y^2 + z^2)$]

3. Prove that $\iiint \frac{dx\, dy\, dz}{\sqrt{1 - x^2 - y^2 - z^2}} = \frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real.

4. *Evaluate* $\iiint \sqrt{\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}}\, dx\, dy\, dz$, *integral being taken over all positive values of* x, y, z *such that* $x^2 + y^2 + z^2 \leq 1$.