UNIT-2
(Differential Calculus-I)

LECTURE 13

Introduction of Successive Differentiation, nth derivative of some elementary functions

Topics:

- Successive Derivatives
- Nth Derivative of standard functions
- Leibnitz Theorem
- Partial Derivative
- Total Derivative
- Euler’s theorem on Homogenous function
- Curve Tracing

Definitions and Notations

Successive differentiation is the differentiation of a function successively to derive its higher order derivatives.

If $y = f(x)$ be a function of x, then the derivative (or differential coefficient) of y w. r. t. x is denoted by $\frac{dy}{dx}$ or Dy or $f'(x)$ or y_1 and this is called the first derivative of y w. r. t. x, where $D \equiv \frac{d}{dx}$.

If $\frac{dy}{dx}$ can be differentiated once again, i.e., if $y = f(x)$ is derivable twice w. r. t. x, then the derivative of $\frac{dy}{dx}$ w. r. t. x is denoted by $\frac{d^2y}{dx^2}$ or D^2y or $f''(x)$ or y_2 and this is called the second derivative of y w. r. t. x.

Similarly, if $\frac{d^2y}{dx^2}$ can be differentiated once again, i.e., if $y = f(x)$ is derivable thrice w. r. t. x, then the derivative of $\frac{d^2y}{dx^2}$ w. r. t. x is denoted by $\frac{d^3y}{dx^3}$ or D^3y or $f'''(x)$ or y_3 and this is called the third derivative of y w. r. t. x.

In a similar manner, we can find the fourth derivative, fifth derivative and, in general, the nth derivative of y w. r. t. x by differentiating successively the given function y w. r. t. x four times, five times and n times.

Following notations are generally used for the successive derivatives of y w. r. t. x :

	First derivative	Second derivative	Third derivative	....	n^th derivative
or	y_1	y_2	y_3	....	y_n
or	$f'(x)$	$f''(x)$	$f'''(x)$	....	$f^n(x)$
or	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	....	$\frac{d^ny}{dx^n}$
or	Dy	D^2y	D^3y	....	D^ny

n^{th} DIFFERENTIAL COEFFICIENT OF STANDARD FUNCTION:

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1. n^{th} Derivative of $(ax+b)^m$,

Let

$$y = (ax+b)^m$$

$$y_1 = ma\,(ax+b)^{m-1}$$

$$y_2 = m\,(m-1)a^2\,(ax+b)^{m-2}$$

....

....

$$y_n = m(m-1)\,(m-2)\,\ldots\,(m-\overline{n-1})\,a^n\,(ax+b)^{m-n}$$

Case I. When m is positive integer, then

$$y_n = \frac{m\,(m-1)\ldots(m-n+1)(m-n)\ldots3\cdot2\cdot1}{(m-n)\ldots3\cdot2\cdot1}\,a^n(ax+b)^{m-n}$$

$$\Rightarrow \qquad \boxed{y_n = \frac{d^n}{dx^n}(ax+b)^m = \frac{\lfloor m}{\lfloor m-n}a^n(ax+b)^{m-n}}$$

Case II. When $m = n =$ +ve integer

$$y_n = \frac{\lfloor n}{\lfloor 0} a^n (ax+b)^0 = \lfloor n \, a^n \Rightarrow \boxed{\frac{d^n}{dx^n}(ax+b)^n = \lfloor n \, a^n}$$

Case III. When $m = -1$, then

$$y = (ax+b)^{-1} = \frac{1}{(ax+b)}$$

$$\therefore \quad y_n = (-1)(-2)(-3)\ldots(-n)\, a^n\, (ax+b)^{-1-n}$$

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\left\{\frac{1}{ax+b}\right\} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}}$$

Case IV. Logarithm case: When $y = \log(ax+b)$, then

$$y_1 = \frac{a}{(ax+b)} = a(ax+b)^{-1} = \frac{a(0!)}{(ax+b)}$$

$$y_2 = \frac{-a^2.1}{(ax+b)^2} = -\frac{a^2.(1!)}{(ax+b)^2}.$$

$$y_3 = \frac{a^3.2}{(ax+b)^3} = \frac{a^3.(2!)}{(ax+b)^3}.$$

$$y_4 = \frac{-a^4 2.3}{(ax+b)^4} = (-1)^3.\frac{a^4.(3!)}{(ax+b)^4}$$ and so on.

In general, $$y_n = (-1)^{n-1}.\frac{a^n.(n-1)!}{(ax+b)^n}$$

Hence $$D^n \log(ax+b) = (-1)^{n-1}.\frac{a^n.(n-1)!}{(ax+b)^n}.$$

Note: $$D^n \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

2. Exponential Function

(i) Consider $y = a^{mx}$

$$y_1 = ma^{mx}. \log_e a$$

$$y_2 = m^2a^{mx} (\log_e a)^2$$

....

....

$$\boxed{y_n = m^n a^{mx} (\log_e a)^n}$$

(ii) Consider $y = e^{mx}$

Putting $a = e$ in above $\boxed{y_n = m^n e^{mx}}$

Example. Find the nth derivatives of $\dfrac{1}{1-5x+6x^2}$.

Solution.

$$\text{Let } y=\frac{1}{6x^2-5x+1}=\frac{1}{(2x-1)(3x-1)}.$$

$$\therefore \frac{1}{6x^2-5x+1}\equiv\frac{A}{2x-1}+\frac{B}{3x-1}\equiv\frac{A(3x-1)+B(2x-1)}{(2x-1)(3x-1)},$$

$$\therefore 1 = A(3x-1)+B(2x-1)$$

Putting $x=\dfrac{1}{3}, 1=-\dfrac{B}{3}$, i.e. $B=-3$; putting $x=\dfrac{1}{2}, A=2.$

$$\text{Hence } y=\frac{2}{2x-1}-\frac{3}{3x-1}=2(2x-1)^{-1}-3(3x-1)^{-1}$$

Therefore $y_n = \dfrac{d^n}{dx^n}\left[2(2x-1)^{-1}\right] - \dfrac{d^x}{dx^n}\left[3(3x-1)^{-1}\right]$

Now we apply the formula,

$$D^n(ax+b)^{-1} = (-1)^n(n!)(ax+b)^{-n-1}a^n.$$

Hence $y_n = 2.2^n(-1)^n(n!)(2x-1)^{-n-1} - 3.3^n(-1)^n(n!)(3x-1)^{-n-1}$.

or $y_n = (-1)^n(n!)\left[\dfrac{2^{n+1}}{(2x-1)^{n+1}} + \dfrac{3^{n+1}}{(3x-1)^{n+1}}\right]$.

Example *Find the n^{th} differential co-efficient of log $(ax + x^2)$.*

Sol. Let
$$y = \log(ax + x^2) = \log x(a + x)$$
$$= \log x + \log(x + a)$$

Differentiating n times,

$$y_n = \frac{d^n}{dx^n}\log x + \frac{d^n}{dx^n}\log(x + a)$$

$$= \frac{(-1)^{n-1}(n-1)!\,.\,1^n}{x^n} + \frac{(-1)^{n-1}(n-1)!\,.\,1^n}{(x+a)^n}$$

$$= (-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right].$$

$$\frac{d^n}{dx^n}\{\log(ax+b)\} = \frac{(-1)^{n-1}\,\underline{|(n-1)}\,a^n}{(ax+b)^n}$$

n^{th} Derivative of $y = \sin(ax + b)$

If $y = \sin(ax+b)$, then

$$y_1 = a\cos(ax+b) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

$$y_2 = a^2\cos(ax+b) = a^2\sin\left[\frac{2\pi}{2} + (ax+b)\right]$$, and so on.

In General, $y_n = a^n \sin\left(ax + b + \frac{1}{2}n\pi\right)$.

Hence, $D^n \sin(ax+b) = a^n \sin\left[ax + b + \frac{1}{2}n\pi\right]$

Note: $D^n \sin x = \sin\left[x + \left(\frac{n\pi}{2}\right)\right]$

n^{th} Derivative of $y = \cos(ax + b)$

If $y = \cos(ax+b)$, then

$$y_1 = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = -a^2\sin\left(\frac{\pi}{2}+ax+b\right) = a^2\cos\left(\frac{2\pi}{2}+ax+b\right)$$, and so on

In general, $y_n = a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$.

Hence, $D^n\cos x(ax+b) = a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$.

Note : $D^n\cos x = \cos\left(x+\frac{1}{2}n\pi\right)$.

Example 1 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let $y = \sin 6x \cos 4x$

$$= \frac{1}{2}(\sin 10x + \sin 2x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

Example 2 Find n^{th} derivative of sin^2xcos^3x

Solution: Let y = sin^2xcos^3x

$$\because \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$$

$$= sin^2xcos^2x\ cosx$$

$$= \frac{1}{4} sin^2 2x\ cosx = \frac{1}{8}(1 - cos4x)cosx$$

$$= \frac{1}{8} cosx - \frac{1}{8} cos4x\ cosx$$

$$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$$

$$= \frac{1}{16}(2cosx - cos3x - cos5x)$$

$$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

Example 3 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x = (sin^2x)^2$

$$= \left(\frac{1}{2}2\,sin^2x\right)^2$$

$$= \frac{1}{4}((1 - \cos 2x)^2$$

$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$$

$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$$

$$= \frac{3}{8} - \frac{1}{2}cos2x + \frac{1}{8}cos4x$$

$$\therefore y_n = -\frac{1}{2}2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n cos\left(4x + \frac{n\pi}{2}\right)$$

n^{th} Derivative of $y = e^{ax}\sin(bx + c)$

Consider the function $y = e^{ax} \sin (bx + c)$

$$y_1 = e^{ax}.b \cos (bx + c) + ae^{ax} \sin (bx + c)$$

$$= e^{ax} [b \cos (bx + c) + a \sin (bx + c)]$$

To rewrite this in the form of sin, put

$$a = r \cos \phi, b = r \sin \phi, \text{ we get}$$

$$y_1 = e^{ax} [r \sin \phi \cos (bx + c) + r \cos \phi \sin (bx + c)]$$

$$y_1 = re^{ax} \sin (bx + c + \phi)$$

Here, $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$

Differentiating again w.r.t. x, we get

$$y_2 = rae^{ax} \sin (bx + c + \phi) + rbe^{ax} \cos (bx + c + \phi)$$

Substituting for a and b, we get

$$y_2 = re^{ax}.\, r\cos\phi \sin(bx + c + \phi) + re^{ax}\, r\sin\phi\cos(bx + c + \phi)$$

$$y_2 = r^2e^{ax}\,[\cos\phi\sin(bx + c + \phi) + \sin\phi\cos(bx + c + \phi)]$$

$$= r^2 e^{ax} \sin(bx + c + \phi + \phi)$$

$\therefore$ $$y_2 = r^2 e^{ax} \sin(bx + c + 2\phi)$$

Similarly, $$y_3 = r^3e^{ax} \sin(bx + c + 3\phi)$$

....

$$y_n = \frac{d^n}{dx^n} e^{ax}\sin(bx + c) = r^n e^{ax} \sin(bx + c + n\phi)$$

$$\therefore y_n = e^{ax}\,(a^2 + b^2)^{\frac{n}{2}} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

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n^{th} Derivative of $y = e^{ax}\cos\ (bx + c)$

Similarly if $y = e^{ax}\cos(ax + b)$

$$y_n = e^{ax}\ r^n\ \cos(bx + c + n\alpha)$$

$$= e^{ax}\ (a^2 + b^2)^{\frac{n}{2}}\ \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & m>0, m>n \\ 0, & m>0, \quad m<n, \\ n!\, a^n, & m=n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, & m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$
$y = e^{ax}\sin(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
$y = e^{ax}\cos(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example :- If $y = \frac{x^n - 1}{x - 1}$ then find y_n.

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Solution :-

$$y = \frac{x^n - 1}{x - 1}$$

$$y = x^{n-1} + x^{n-2} + \cdots + x + 1$$

$$y_n = 0$$

Example :- Find the nth derivative of $\sin^2 x \cos^3 x$.

Solution :- $y = \sin^2 x \cos^3 x$

$$y = \sin^2 x \cos^2 x \cos x = \frac{1}{4}\sin^2 2x \cos x$$

$$= \frac{1}{4} \times \frac{1}{2}(1 - \cos 4x)\cos x \qquad \because \sin^2 2x = \frac{1}{2}(1-\cos 4x)$$

$$= \frac{1}{8}\left(\cos x - \frac{2\cos 4x \cos x}{2}\right)$$

$$= \frac{1}{8}\cos x - \frac{1}{16}\left(\cos 5x + \cos 3x\right)$$

$$y = \frac{1}{8}\cos x - \frac{1}{16}\cos 5x - \frac{1}{16}\cos 3x$$

then

$$y_n = \frac{1}{8}\cos\left(x + \frac{n\pi}{2}\right) - \frac{5^n}{16}\cos\left(5x + \frac{n\pi}{2}\right) - \frac{3^n}{16}\cos\left(3x + \frac{n\pi}{2}\right)$$

Example 4 Find the n^{th} derivative of $e^{3x}\cos x \sin^2 2x$

Solution: Let $y = e^{3x}\cos x \sin^2 2x$

$$\text{Now } \cos x \sin^2 2x = \frac{1}{2}\left(\cos x - \cos x \cos 4x\right)$$

$$\because \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$$

$$= \frac{1}{2}\left(\cos x - \frac{1}{2}\left(\cos 5x + \cos 3x\right)\right)$$

$$\Rightarrow y = e^{3x}\cos x \sin^2 2x = \frac{1}{2}e^{3x}\cos x - \frac{1}{4}e^{3x}\cos 5x - \frac{1}{4}e^{3x}\cos 3x$$

$$\boxed{\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]}$$

$$\therefore y_n = \frac{1}{2}e^{3x}(9+1)^{\frac{n}{2}}\cos\left(x+n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}(9+25)^{\frac{n}{2}}\cos\left(5x+n\tan^{-1}\frac{5}{3}\right)$$

$$-\frac{1}{4}e^{3x}(9+9)^{\frac{n}{2}}\cos\left(3x+n\tan^{-1}\frac{3}{3}\right)$$

$$= \frac{1}{2}e^{3x}10^{\frac{n}{2}}\cos\left(x+n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}34^{\frac{n}{2}}\cos\left(5x+n\tan^{-1}\frac{5}{3}\right)$$

$$-\frac{1}{4}e^{3x}18^{\frac{n}{2}}\cos(3x+n\tan^{-1}1)$$

$y = e^{ax}\cos(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 5 If $y = \sin ax + \cos ax$, prove that $y_n = a^n[1+(-1)^n \sin 2ax]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$$\therefore y_n = a^n\left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$$

$$= a^n\left[\left\{\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= a^n\left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2\sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= a^n[1+\sin(2ax+n\pi)]^{\frac{1}{2}}$$

$$= a^n[1+\sin 2ax \cos n\pi + \cos 2ax \sin n\pi]^{\frac{1}{2}}$$

$$= a^n[1+(-1)^n \sin 2ax]^{\frac{1}{2}} \quad \because \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0$$

Example1. If $y = x \log \frac{x-1}{x+1}$. Show that

$$y_n = (-1)^{n-2} \lfloor n-2 \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n} \right]$$

Sol. We have $y = x \log \frac{x-1}{x+1} = x\,[\log(x-1) - \log(x+1)]$

Differentiating w.r. to 'x', we get

$$y_1 = \log(x-1) - \log(x+1) + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log(x-1) - \log(x+1) + \left(1+\frac{1}{x-1}\right) + \left(-1+\frac{1}{x+1}\right)$$

or $y_1 = \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$

Differentiating $(n-1)$ times with respect to x, we get

$$y_n = \frac{d^{n-1}}{dx^{n-1}}\log(x-1) - \frac{d^{n-1}}{dx^{n-1}}\log(x+1) + \frac{d^{n-1}}{dx^{n-1}}(x-1)^{-1} + \frac{d^{n-1}}{dx^{n-1}}(x+1)^{-1}$$

$$\frac{d^n}{dx^n}\{\log(ax+b)\} = \frac{(-1)^{n-1}\lfloor(n-1)\, a^n}{(ax+b)^n}$$

$$\frac{d^n}{dx^n}\left\{\frac{1}{ax+b}\right\} = \frac{(-1)^n \lfloor n\; a^n}{(ax+b)^{n+1}}$$

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$$y_n = \frac{d^{n-1}}{dx^{n-1}}\log(x-1) - \frac{d^{n-1}}{dx^{n-1}}\log(x+1) + \frac{d^{n-1}}{dx^{n-1}}(x-1)^{-1} + \frac{d^{n-1}}{dx^{n-1}}(x+1)^{-1}$$

$$= \frac{(-1)^{n-2}\lfloor n-2}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}} + \frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n}$$

$$= \frac{(-1)^{n-2}\lfloor n-2}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)\lfloor n-2}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)\lfloor n-2}{(x+1)^n}$$

$$= (-1)^{n-2}\lfloor n-2\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} - \frac{(n-1)}{(x-1)^n} - \frac{(n-1)}{(x+1)^n}\right]$$

$$= (-1)^{n-2}\lfloor n-2\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Practice Questions: 1

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1. If $y = \sin^3 x$, find y_n.

$$\left[\textbf{Ans. } \frac{3}{4}\sin\left(x+n\frac{\pi}{2}\right)-\frac{1}{4}.3^n.\sin\left(3x+n\frac{\pi}{2}\right)\right]$$

2. Find the n^{th} derivative of $\cos x \cos 2x \cos 3x$

$$\text{Ans.}\frac{1}{4}\left[2^n\cos\left(2x+\frac{n\pi}{2}\right)+4^n\cos\left(4x+\frac{n\pi}{2}\right)+6^n\cos\left(6x+\frac{n\pi}{2}\right)\right]$$

3. If $y = e^x \cos^3 x$, find y_n i.e. the n^{th} derivative of y

$$\text{Ans. } \frac{3}{4}\, 2^{\frac{n}{2}} e^x \cos\left(x+n\frac{\pi}{4}\right)+\frac{1}{4}.10^{\frac{n}{2}}\, e^x \cos(3x+n\tan^{-1}3)$$

4. If $y = \cos^{-1} x$, prove that $(1-x^2)y_2 - xy_1 = 0$

Practice Questions: 2

1. Find y_n, when $y = \dfrac{1-x}{1+x}$. $\left[\textbf{Ans. } \dfrac{2(-1)^n \lfloor n}{(x+1)^{n+1}}\right]$

2. Find nth derivative of $\log x^2$. $\left[\textbf{Ans. } (-1)^{n-1} \lfloor n-1 \cdot 2x^{-n}\right]$

3. Find the n^{th} derivative of $\dfrac{x^4}{(x-1)(x-2)}$

   Ans. $(-1)^n n! \left[\dfrac{16}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right]$

4. Find n^{th} differential coefficient of $y = \log[(ax+b)(cx+d)]$

   Ans. $y_n = (-1)^{n-1}(n-1)! \left[\dfrac{a^n}{(ax+b)^n} + \dfrac{c^n}{(cx+d)^n}\right]$

LECTURE 14

Leibnitz's Theorem &

nth derivative of product of functions

LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \cdots + {}^nC_r\, u_{n-r}\, v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

or

$$D^n(uv) = u_n v + n\, u_{n-1}\, v_1 + \frac{n(n-1)}{2!}\, u_{n-2}\, v_2 + \ldots + {}^nC_r\, u_{n-r}\, v_r + \cdots + u v_n$$

Example . Find the nth derivative of $x^2 \sin 3x$.

Sol. Let $u = \sin 3x$ and $v = x^2$

$$\therefore \quad D^n(u) = D^n(\sin 3x) = 3^n \sin\left(3x + \frac{n\pi}{2}\right)$$

$$D(v) = 2x,\ D^2(v) = 2,\ D^3(v) = 0$$

By Leibnitz's theorem, we have

$$D^n(u.v) = {}^nc_0\, D^n(u).v + {}^nc_1 D^{n-1}(u).\, D(v) + {}^nc_2\, D^{n-2}(u).D^2(v) + \ldots + {}^nc_n\, u.\, D^n v$$

$$D^n(x^2 \sin 3x) = D^n(\sin 3x)x^2 + {}^nc_1\, D^{n-1}(\sin 3x) \cdot D(x^2) + {}^nc_2\, D^{n-2}(\sin 3x) \cdot D^2(x^2)$$

$$D^n(x^2 \sin 3x) = D^n(\sin 3x)x^2 + {}^nc_1\, D^{n-1}(\sin 3x)\cdot D(x^2) + {}^nc_2\, D^{n-2}(\sin 3x)\cdot D^2(x^2)$$

$$= 3^n \sin\left(3x+\frac{n\pi}{2}\right)\cdot x^2 + n3^{n-1}\sin\left(3x+\frac{\overline{n-1}\pi}{2}\right)\cdot 2x$$

$$+\frac{n(n-1)}{\underline{|2}}\cdot 3^{n-2}\sin\left(3x+\frac{\overline{n-2}\pi}{2}\right)\cdot 2$$

$$= 3^n x^2 \sin\left(3x+\frac{n\pi}{2}\right) + 2nx\cdot 3^{n-1}\sin\left(3x+\frac{\overline{n-1}\pi}{2}\right)$$

$$+ 3^{n-2}n(n-1)\cdot\sin\left(3x+\frac{\overline{n-2}\pi}{2}\right).$$

Example . Find the nth derivative of $e^x \log x$.

Sol. Let $u = e^x$ and $v = \log x$

Then $D^n(u) = e^x$ and $D^n(v) = \dfrac{(-1)^{n-1}\lfloor n-1}{x^n}$

$$D^n \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

By Leibnitz's theorem, we have

$$D^n(e^x \log x) = D^n e^x \log x + {}^nc_1 D^{n-1}(e^x) D(\log x) + {}^nc_2 D^{n-2}(e^x) D^2(\log x) + \ldots + e^x D^n(\log x)$$

$$= e^x \cdot \log x + ne^x \cdot \frac{1}{x} + \frac{n(n-1)}{\lfloor 2} e^x \left(-\frac{1}{x^2}\right) + \ldots + e^x \frac{(-1)^{n-1}\lfloor n-1}{x^n}$$

$$\Rightarrow \quad D^n(e^x \log x) = e^x \left[\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \ldots + \frac{(-1)^{n-1}\lfloor n-1}{x^n}\right].$$

Example. 3.

If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

and $x^2 y_{n+2} + (2n-1)xy_{n+1} + (n^2+1)y_n = 0.$

Solution.

Let $y = a\cos(\log x) + b\sin(\log x)$,

$$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x)\frac{1}{x} \text{ or } xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Now again differentiating both sides, we get

$$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

or $x^2 y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or $x^2 y_2 + xy_1 = -y$

or $x^2 y_2 + xy_1 + y = 0.$

Again differentiating both sides n times by Leibnitz's theorem,

$$D^n(x^2y_2)+D^n(xy_1)+D^n(y)=0.$$

$$\text{or } \left[x^2D^ny_2+nDx^2D^{n-1}y_2+\frac{n(n-1)}{2}D^2x^2D^{n-2}y_2\right]+xD^ny_1+nD^{n+1}y_1+y_n=0$$

$$\text{or } x^2y_{n+2}+2nxy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n+y_n=0$$

$$\text{or } x^2y_{n+2}+(2n-1)xy_{n+1}+(n^2+1)y_n=0.$$

Example 4. If $y = \sin(m \sin^{-1} x)$. prove that

$(1-x^2)y_2 - xy_1 + m^2 y = 0$ and deduce that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0.$$

Solution: Let $y = \sin(m \sin^{-1} x)$.

$$y_1 = \cos(m \sin^{-1} x).\frac{m}{\sqrt{(1-x^2)}}. \quad \text{or} \quad (1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x).$$

$$or\ (1-x^2)y_1^2 = m^2 - m^2 \sin^2(m \sin^{-1} x) = m^2 - m^2 y^2$$

$$\therefore\ (1-x^2)y_1^2 + m^2 y^2 = m^2.$$

Again differentiating both sides, we have

$$2y_1y_2(1-x^2)-2xy_1^2+2m^2yy_1=0. \text{ or } y_2(1-x^2).-\ xy_1+m^2y=0.$$

Now differntiating n time by Leibnitz's theorem, we get

$$\left[(1-x^2)\,y_{n+2}+n\,(-2x)\,y_{n+1}+\frac{n(n-1)}{2}(-2)\,y_n\right]-xy_{n+1}-n(1)y_n+m^2y_n=0,$$

$$\text{or } (1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0.$$

Example 5. If $y = \log\left(x + \sqrt{1 + x^2}\right)$

Prove that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log\left(x + \sqrt{1 + x^2}\right)$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1 + x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1 + x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1 + x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\left[y_{n+2}(1+x^2) + n_{C_1} y_{n+1} 2x + n_{C_2} y_n .\, 2\right] + \left(y_{n+1} x + n_{C_1} y_n .\, 1\right) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1} 2nx + n(n-1) y_n + y_{n+1} x + n y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + (2n+1) x y_{n+1} + n^2 y_n = 0$$

Example 6. If $y = x^n \log x$, then prove that

(i) $y_{n+1} = \dfrac{\lfloor n}{x}$ (ii) $y_n = ny_{n-1} + \lfloor (n-1)$.

Sol. (i) We have $y = x^n \log x$

Differentiating w.r.t. to x, we get

$$y_1 = nx^{n-1} \cdot \log x + \frac{x^n}{x}$$

$$\Rightarrow \quad xy_1 = nx^n . \log x + x^n$$

$$xy_1 = ny + x^n \qquad \ldots(i)$$

Differentiating equation (i) n times, we get

$$xy_{n+1} + ny_n = ny_n + \lfloor n$$

$$\Rightarrow \quad y_{n+1} = \frac{\lfloor n}{x} \quad \text{Proved.}$$

(ii) $$y_n = \frac{d^n}{dx^n}\left(x^n . \log x\right) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx} x^n . \log x\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(\frac{x^n}{x} + n x^{n-1} . \log x\right)$$

$$= n \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1} . \log x\right) + \frac{d^{n-1}}{dx^{n-1}} . x^{n-1}$$

$$= n y_{n-1} + |\underline{(n-1)}. \quad \textbf{Proved.}$$

As $y_n = \dfrac{d^n}{dx^n}\left(x^n \log x\right)$

$\therefore \; y_{n-1} = \dfrac{d^{n-1}}{dx^{n-1}}\left(x^{n-1} \log x\right)$

Example :- Find the nth derivative of $x^{n-1} \log x$.

Solution :- $y = x^{n-1} \log x$ (base e)

$$y_1 = (n-1)x^{n-2} \log x + x^{n-2}$$

multiplying both sides by x then we get

$$x y_1 = (n-1) x^{n-1} \log x + x^{n-1}$$

$$x y_1 = (n-1) y + x^{n-1}$$

...

Now differentiating both sides $(n-1)$ times by Leibnitz theorem, we get

$$x\, y_n + (n-1)\, y_{n-1} = (n-1)\, y_{n-1} + \lfloor n-1$$

$$\boxed{y_n = \frac{\lfloor n-1}{x}}$$

Practice Questions

1. If $y = \cos(m \log x)$, show that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (m^2 + n^2)y_n = 0$

2. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$

3. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

4. If $y = [x - \sqrt{x^2 - 1}]^m$, show that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Leibnitz theorem

$$D^n(u.v) = v\,D^n(u) + {}^nC_1\,D^{n-1}(u).D(v) + {}^nC_2\,D^{n-2}(u).D^2(v) + \cdots + u.D^n(v)$$

or

$$D^n(u.v) = v\,u_n + {}^nC_1\,u_{n-1}v_1 + {}^nC_2\,u_{n-2}v_2 + \cdots + u\,v_n$$

Examples

Example :- If $y = e^{m\cos^{-1}x}$ then find the relation between y_n, y_{n+1} and y_{n+2}. [2015, 19]

Solution :-

$$y = e^{m\cos^{-1}x} \quad \cdots (1)$$

Differentiating (1) w.r.to x

$$y_1 = e^{m\cos^{-1}x} \cdot \left(\frac{-m}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2} \cdot y_1 = -my$$

Squaring both sides, we get

$$(1-x^2)y_1^2 = m^2y^2 \quad \cdots (2)$$

Differentiating (2) w. r. to x

$$(-2x)\,y_1^2 + 2(1-x^2)\,y_1 y_2 = 2m^2 y\, y_1$$

which gives

$$(1-x^2)\,y_2 - x y_1 - m^2 y = 0 \qquad \cdots (3)$$

Now differentiating both sides n times w.r. to x by Leibnitz theorem

$$D^n\left[(1-x^2)\,y_2\right] - D^n\left[x y_1\right] - m^2 D^n y = 0$$

$$(1-x^2)y_{n+2} + {}^{n}C_1\, y_{n+1}(-2x) + {}^{n}C_2\, y_n(-2)$$

$$- x y_{n+1} - {}^{n}C_1\, y_n - m^2 y_n = 0$$

$$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - \frac{2n(n-1)}{2} y_n - x y_{n+1} - n y_n$$

$$- m^2 y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} - n^2 y_n + n y_n - n y_n$$

$$- m^2 y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} - (n^2 + m^2) y_n = 0$$

Example :- If $y = \left(\frac{1+x}{1-x}\right)^{1/2}$, prove that

$$(1-x^2)\, y_n - [2(n-1)x + 1]\, y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0$$

[2011.]

Solution :-

Given function

$$y = \left(\frac{1+x}{1-x}\right)^{1/2}$$

Taking logrithm on both sides with base e.

$$\ln y = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right] \quad \cdots (1)$$

Differentiating (1) w.r. to x, we get

$$\frac{1}{y}\cdot y_1 = \frac{1}{2}\left[\frac{1}{1+x}+\frac{1}{1-x}\right] = \frac{1}{2}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right]$$

$$(1-x^2)y_1 = y \qquad \cdots (2)$$

Diff. (2) $(n-1)$ times w.r. to x. by Leibnitz theorem

$$(1-x^2)\,y_n + (n-1)\,y_{n-1}(-2x) + \frac{(n-1)(n-2)}{2}\,y_{n-2}(-2) = y_{n-1}$$

$$(1-x^2)\,y_n - [2(n-1)x+1]\,y_{n-1} - (n-1)(n-2)\,y_{n-2} = 0$$

Example :- If $y = \sin \log (x^2+2x+1)$, prove that

$$(1+x)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$$

[2012, 18]

Solution :- Hint $y = \sin \log (x^2+2x+1)$

Diff. w. r. to x

$$y_1 = \cos \log (x^2+2x+1) \left(\frac{1}{x^2+2x+1}\right)(2x+2)$$

$$y_1 = \cos [\log (x^2+2x+1)] \left(\frac{2}{x+1}\right)$$

$$(x+1)\, y_1 = 2 \cos [\log (x^2+2x+1)] \quad \cdots (1)$$

Diff. again w.r. to x

$$y_1 + (x+1)y_2 = -2\sin[\log(x^2+2x+1)]\frac{d}{dx}\log(x^2+2x+1)$$

$$= -2y \cdot \frac{1}{x^2+2x+1} \cdot (2x+2)$$

$$(x+1)y_2 + y_1 = -4\frac{y}{(x+1)}$$

$$(x+1)^2 \cdot y_2 + (x+1)y_1 + 4y = 0 \quad \cdots (2)$$

Now by using Leibnitz theorem, diff. (2) n times w.r. to x, we get

$$(1+x)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

Example If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$

[2014]

Sol. Given $$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$\Rightarrow$ $$y^{2/m} - 2xy^{1/m} + 1 = 0$$

or $$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$\Rightarrow$ $$z^2 - 2xz + 1 \qquad (y^{1/m} = z)$$

$\therefore$ $$z = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$\Rightarrow$ $$y^{1/m} = x \pm \sqrt{x^2 - 1} \Rightarrow y = \left[x \pm \sqrt{x^2 - 1}\right]^m \quad ...(i)$$

Differentiating equation (i) w.r.t. x, we get

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1}\left[1 \pm \frac{2x}{2\sqrt{x^2-1}}\right] = \frac{m\left[x \pm \sqrt{x^2-1}\right]^m}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{x^2-1}} \Rightarrow y_1\sqrt{x^2-1} = my$$

or $$y_1^2(x^2-1) = m^2y^2 \qquad ...(ii)$$

Differentiating both sides equation (*ii*) w.r.t. x, we obtain

$$2y_1y_2(x^2-1) + 2xy_1^2 = 2m^2\,yy_1$$

$$\Rightarrow \quad y_2(x^2-1) + xy_1 - m^2y = 0$$

Differentiating n times by Leibnitz's theorem w.r.t. x, we get

Example :- If $I_n = \frac{d^n}{dx^n}(x^n \log x)$

then show that $I_n = n I_{n-1} + \lfloor n-1$

[2016]

Solution :-

$$I_n = \frac{d^n}{dx^n}(x^n \log x)$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dx}(x^n \log x)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[n x^{n-1} \log x + \frac{x^n}{x}\right]$$

$$= n \frac{d^{n-1}}{dx^{n-1}}(x^{n-1} \log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$I_n = n I_{n-1} + \lfloor n-1$$

Practice Questions

1. If $y = e^{\tan^{-1}x}$, prove that

$$(1+x^2)\,y_{n+2} + [(2n+2)x - 1]\,y_{n+1} + n(n+1)\,y_n = 0$$

[2013, 2017, 2020]

2. If $y\sqrt{x^2-1} = \log_e\left(x+\sqrt{x^2-1}\right)$, prove that $(x^2-1)y_{n+1} + (2n+1)xy_n + n^2y_{n-1} = 0$.

2022-23

Practice Questions

3. If $y = x\cos(\log x)$, prove that

$$x^2 y_{n+2} + (2n-1)xy_{n+1} + (n^2 - 2n + 2)y_n = 0$$

4. If $y = (\sin^{-1} x)^2$, prove that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2)y_n = 0$$

LECTURE-15

To find n^{th} derivative of a function at x=0

DETERMINATION OF THE VALUE OF THE *n*TH DERIVATIVE OF A FUNCTION FOR $x = 0$

Some times we have to find out the nth derivative of a function at $x = 0$.

Working Rule:

Step 1. Write down the given function as $y = f(x)$

Step 2. Find y_1

Step 3. Find y_2

Step 4. Differentiate n times both the sides of the equation obtained in step 3 by Leibnitz theorem.

Step 5. Substitute $x = 0$ in y, y_1, y_2 and the nth derivative obtained in steps 1, 2, 3 and 4.

Step 6. Put the values of $y(0), y_1(0), y_2(0)$ in the result obtained in step 5.

Step 7. Put $n = 1, 2, 3, 4$ in the last equation of step 6.

Step 8. Find out $y_n(0)$ when n is even and n is odd.

Leibnitz theorem

$$D^n(u.v) = v\,D^n(u) + {}^nc_1\,D^{n-1}(u).D(v) + {}^nc_2\,D^{n-2}(u).D^2(v) + \cdots + u.D^n(v)$$

or

$$D^n(u.v) = v\,u_n + {}^nc_1\,u_{n-1}v_1 + {}^nc_2\,u_{n-2}v_2 + \cdots + u\,v_n$$

Example :- If $y = x^2 \exp(2x)$, determine $(y_n)_0$

Solution :-

$$y = x^2 e^{2x} \quad \cdots (1)$$

Diff (1) n times w.r.to x by Leibnitz theorem

$$y_n = 2^n e^{2x} . x^2 + {}^nc_1 . 2^{n-1} e^{2x} . 2x + {}^nc_2 \, 2^{n-2} e^{2x} . 2$$

$$y_n = 2^n x^2 e^{2x} + 2^n n x e^{2x} + (n^2 - n) 2^{n-2} e^{2x} \quad \cdots (2)$$

by putting $x=0$ in (2) we get

$$(y_n)_{x=0} = (n^2 - n) 2^{n-2}$$

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Example :- If $y = \sin(a\sin^{-1}x)$, find $(y_n)_0$. [2015, 2018, 2020] **2022-23**

Solution :- $y = \sin(a\sin^{-1}x)$ ...(1)

$$y_1 = \cos(a\sin^{-1}x)\left(\frac{a}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\, y_1 = a\cos(a\sin^{-1}x) \quad \text{---(2)}$$

squaring both sides,

$$(1-x^2)\, y_1^2 = a^2 \cos^2(a\sin^{-1}x)$$

$$= a^2\left(1-\sin^2(a\sin^{-1}x)\right)$$

$$(1-x^2)\, y_1^2 = a^2(1-y^2) \quad \cdots(3)$$

$$(1-x^2)\, y_1^2 = a^2 \cos^2(a\sin^{-1}x)$$

$$= a^2\left(1-\sin^2(a\sin^{-1}x)\right)$$

$$(1-x^2)\, y_1^2 = a^2(1-y^2) \quad \cdots(3)$$

diff (3) w.r.to x

$$(-2x)\, y_1^2 + 2y_1 y_2(1-x^2) = -2a^2 y y_1$$

$$(1-x^2)\, y_2 - x y_1 + a^2 y = 0 \quad \cdots(4)$$

diff. (4) n times w.r.to x by using Leibnitz theorem, we get

$$(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - (n^2-a^2)\, y_n = 0 \quad \cdots(5)$$

Now we will find $y_n(0)$,

by putting $x=0$ in equations (1), (2), (3)

(4) and (5) we get,

$(y)_0 = 0$

$(y_1)_0 = a$

$(y_2)_0 = 0$

and $(y_{n+2})_0 = (n^2 - a^2)\, y_n(0) \quad \text{---} (6)$

putting $n = 1, 2, 3, \ldots$ in (6) we get

$y_3(0) = (1^2 - a^2)\, y_1(0) = (1^2 - a^2)a$

$y_4(0) = (2^2 - a^2)\, y_2(0) = 0$

$y_5(0) = (3^2 - a^2)\, y_3(0) = (3^2 - a^2)(1^2 - a^2)a$

$y_6(0) = (4^2 - a^2)\, y_4(0) = 0$

In general

$$y_n(0) = \begin{cases} 0 & \text{when } n \text{ is even} \\ [(n-2)^2 - a^2][(n-4)^2 - a^2] \cdots (3^2 - a^2)(1^2 - a^2)a & \text{when } n \text{ is odd} \end{cases}$$

Example If $y = (sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ......②

Squaring both the sides, we get

$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1-x^2){y_1}^2 = 4\,y$

Differentiating the above equation w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0$ ......③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{c_1} y_{n+1}(-2x) + n_{c_2} y_n(-2)] - \left(y_{n+1}x + n_{c_1} y_n 1\right) = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_n n^2 = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$$y(0) = 0\,, y_1(0) = 0 \text{ and } y_2(0) = 2$$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.\,2^2.\,4^2 \ldots\ldots\ldots.(n-2)^2, & if\ n\ is\ even \end{cases}$$

n is not 2

Example If $y = e^{m\sin^{-1}x}$, show that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. Also find $y_n(0)$.

Solution: Here $y = e^{m\sin^{-1}x}$ ...①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}} e^{m\sin^{-1}x}$$

$$= \frac{my}{\sqrt{1-x^2}} \quad ......②$$

$$\Rightarrow (1-x^2){y_1}^2 = m^2y^2$$

Differentiating above equation w.r.t. x, we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0 \quad ......③$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2)+n_{C_1}y_{n+1}(-2x)+n_{C_2}y_n(-2)]-\left(y_{n+1}x+n_{C_1}y_n 1\right)-m^2y_n=0$$

$$\Rightarrow y_{n+2}(1-x^2)-y_{n+1}2nx-n(n-1)y_n-(y_{n+1}x+ny_n)-m^2y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0 \ldots\ldots④$$

To find $y_n(0)$: Putting $x=0$ in ①, ② and ③

$y(0)=1,\ y_1(0)=m$ and $y_2(0)=m^2$

Also putting $x=0$ in , we get

$$y_{n+2}(0)=(n^2+m^2)y_n(0)$$

Putting $n=1,2,3\ldots$ in the above equation, we get

$$y_3(0) = (1^2 + m^2)y_1(0)$$

$$=(1^2 + m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 + m^2)y_2(0)$$

$$= m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$$

$$y_5(0) = \left(3^2 + m^2\right)y_3(0)$$

$$= m(1^2 + m^2)(3^2 + m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2 + m^2)\ldots[(n-2)^2 + m^2], & if\ n\ is\ even \\ m(1^2 + m^2)(3^2 + m^2)\ldots[(n-2)^2 + m^2], & if\ n\ is\ odd \end{cases}$$

Practice Questions

1. If $y = \sin^{-1} x$, prove that $(1 - x^2)y_{n+2} - (2n + 1)\, x\, y_{n+1} - n^2 y_n = 0$.
   Also find the value of y_n when $x = 0$.

2. If $y = e^{m \cos^{-1} x}$ show that
   $$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + m^2)\, y_n = 0$$
   and calculate $y_n(0)$.

3. If $y = \log\,(x + \sqrt{1 + x^2})$, find the value of y_n at $x = 0$.

4 If $y = \left[\log\left\{x + \sqrt{1 + x^2}\right\}\right]^2$, show that

$y_{n+2}(0) = -n^2 y_n(0)$ hence find $y_n(0)$.

Answers

1. **Ans.** When n is even $y_n(0) = 0$ when n is odd, $y_n(0) = 1^2.3^2.5^2 \ldots (n-2)^2$

2. **Ans.** When n is even, then

$$y_n(0) = m^2 \, e^{m\frac{\pi}{2}} \, (2^2 + m^2)(4^2 + m^2) \ldots\ldots\ldots [(n-2)^2 + m^2]$$

when n is odd, then

$$y_n(0) = -m \, e^{m\frac{\pi}{2}} \, (1^2 + m^2)(3^2 + m^2) \ldots\ldots\ldots [(n-2)^2 + m^2]$$

3. **Ans.** when n is even, $y_n(0) = 0$

when n is odd, $y_n(0) = (-1)^{\frac{n-1}{2}} .1^2 .3^2 .5^2 \ldots\ldots\ldots (n-2)^2$

4. **Ans.** n is odd, $y_n(0), = 0$ n is even $y_n(0) = (-1)^{\frac{n-2}{2}} (n-2)^2 (n-4)^2 \ldots .\, 4^2\, 2^2.2.$

LECTURE-16

Introduction to Partial Derivatives

Introduction to Partial Differentiation:-

If $z = f(x, y)$ be a function of two independent variables x and y, then to differentiate z partial diff. is used.

Partial derivatives of $z = f(x, y)$

(i) First order partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

* To find $\frac{\partial z}{\partial x}$, diff. z with respect to x, taking y constant
* To find $\frac{\partial z}{\partial y}$, diff z w.r.to y, taking x as constant

C(i) Second Order Partial Derivatives –

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or f_{xx} ; $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$ or f_{yx}

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or f_{xy} ; $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ or f_{yy}

Note – (1) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ or $f_{xy} = f_{yx}$

(2) The process of differentiating $y = f(x)$ is called ordinary diff.

(3) If $u = f(x_1, x_2, \text{---}, x_n)$ then first order partial derivatives are $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \text{---}, \frac{\partial u}{\partial x_n}$

Similarly we can find higher order partial derivatives.

Q1. Find the first order partial derivatives of the functions:

(i) $u = y^x$, (ii) $z = \log(x^2+y^2)$, (iii) $z = \cos^{-1}\left(\frac{x}{y}\right)$

Sol (i) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(y^x) = y^x \log y$ (Taking y as constant then use $\frac{d}{dx} a^x = a^x \log a$)

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(y^x) = x y^{x-1}$ (Taking x as constant then use $\frac{d}{dx} x^n = n x^{n-1}$)

(ii) $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \log(x^2+y^2) = \frac{1}{x^2+y^2} \frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{x^2+y^2}(2x+0)$

$= \frac{2x}{x^2+y^2}$

Taking y as constant to find $\frac{\partial z}{\partial x}$

Since $z = \log(x^2+y^2)$ is symmetrical w.r. to x and y (as we get same function after interchanging x and y)

Hence to find $\frac{\partial z}{\partial y}$ interchange x and y.

$$\therefore \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

(iii) $$\frac{\partial z}{\partial x} = -\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y}(1) = \frac{-1}{\sqrt{y^2-x^2}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}} \cdot x\left(-\frac{1}{y^2}\right) = \frac{x}{y\sqrt{y^2-x^2}}$$

Q2. If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$, show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Sol. Differentiating u partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} (\sec^2 x)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$\text{L.H.S.} = \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$\text{L.H.S.} = \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \cdot \sec^2 x + \sin 2y \cdot \sec^2 y + \sin 2z \cdot \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\sin x \cos x \cdot \frac{1}{\cos^2 x} + 2\sin y \cos y \cdot \frac{1}{\cos^2 y} + 2\sin z \cos z \cdot \frac{1}{\cos^2 z}}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z}$$

$$= 2 = \text{R.H.S.}$$

Q-3 If $e^{\frac{-z}{x^2-y^2}} = x-y$ then show that (2016-17)

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2-y^2$$

Sol. Taking log on both sides,

$$\frac{-z}{x^2-y^2} = \log(x-y)$$

$$\Rightarrow -z = (x^2-y^2)\log(x-y)$$

$$\text{or } z = (y^2-x^2)\log(x-y) \quad \text{--- (1)}$$

Differentiate eqn. (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = (0-2x)\log(x-y) + (y^2-x^2)\cdot\frac{1}{x-y}(1-0)$$

$$\therefore y\frac{\partial z}{\partial x} = -2xy\log(x-y) + \frac{y(y^2-x^2)}{x-y} \quad \text{--- (2)}$$

and $\dfrac{\partial z}{\partial y} = (2y-0)\log(x-y) + (y^2-x^2)\cdot\dfrac{1}{x-y}(0-1)$

$\therefore x\dfrac{\partial z}{\partial y} = 2xy\log(x-y) - \dfrac{x(y^2-x^2)}{x-y}$ —— ③

Adding equations ② and ③, we get

$y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = \dfrac{(y-x)(y^2-x^2)}{x-y} = -(y^2-x^2) = x^2-y^2$

Q.4 If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, find the value of $\frac{\partial^2 u}{\partial x \partial y}$. (2017-18)

Sol. $\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - \left[2y \tan^{-1}\frac{x}{y} + y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot x\left(-\frac{1}{y^2}\right)\right]$

$= x \cdot \frac{x^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + x \cdot \frac{y^2}{x^2+y^2}$

$= \frac{x^3 + xy^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$

$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$

$$\therefore \frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}$$

Hence

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Example Find all of the first order partial derivatives for the following functions.

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

(b) $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

Solution

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

Let's first take the derivative with respect to x and remember that

as we do so all the y's will be treated as constants. The partial derivative with respect to x is,

$$f_x(x,y) = 4x^3$$

Now, let's take the derivative with respect to y. In this case we treat all x's as constants and so the first term involves only x's and so will differentiate to zero, just as the third term will. Here is the partial derivative with respect to y.

$$f_y(x,y) = \frac{3}{\sqrt{y}}$$

(b) $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

Here is the partial derivative with respect to x.

$$\frac{\partial w}{\partial x} = 2xy + 43$$

Here is the derivative with respect to y.

$$\frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2(4y)$$

Here is the derivative with respect to z.

$$\frac{\partial w}{\partial z} = -30y^2z^2$$

Example Find all of the first order partial derivatives for the following functions.

(a) $z = \dfrac{9u}{u^2 + 5v}$

(b) $g(x, y, z) = \dfrac{x \sin(y)}{z^2}$

Solution

(a) $z = \dfrac{9u}{u^2 + 5v}$

We also can't forget about the quotient rule.

$$z_u = \frac{9\left(u^2 + 5v\right) - 9u\,(2u)}{\left(u^2 + 5v\right)^2} = \frac{-9u^2 + 45v}{\left(u^2 + 5v\right)^2}$$

$$z_v = \frac{(0)\left(u^2 + 5v\right) - 9u\,(5)}{\left(u^2 + 5v\right)^2} = \frac{-45u}{\left(u^2 + 5v\right)^2}$$

(b) $g\left(x, y, z\right) = \dfrac{x\sin(y)}{z^2}$

$$g_x\left(x,y,z\right) = \frac{\sin(y)}{z^2} \qquad g_y\left(x,y,z\right) = \frac{x\cos(y)}{z^2}$$

$$g\left(x,y,z\right) = x\sin(y)z^{-2}$$

$$g_z\left(x,y,z\right) = -2x\sin(y)z^{-3} = -\frac{2x\sin(y)}{z^3}$$

Example Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ if $u = e^{r\cos\theta}.\cos(r\sin\theta)$

Solution. $u = e^{r\cos\theta}.\cos(r\sin\theta)$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta}.[-\sin(r\sin\theta).\sin\theta] + [\cos\theta.e^{r\cos\theta}].\cos(r\sin\theta)$$

(keeping θ as constant)

$$= e^{r\cos\theta}.[-\sin(r\sin\theta).\sin\theta + \cos(r\sin\theta).\cos\theta]$$

$$= e^{r\cos\theta}.\cos(r\sin\theta + \theta)$$ **Ans.**

$$\frac{\partial u}{\partial \theta} = e^{r\cos\theta}.[-\sin(r\sin\theta).r\cos\theta] + [-r\sin\theta.e^{r\cos\theta}].\cos(r\sin\theta)$$

(keeping r as constant)

$$= -r\,e^{r\cos\theta}.[\sin(r\sin\theta).\cos\theta + \sin\theta\cos(r\sin\theta)]$$

$$= -r\,e^{r\cos\theta}.\sin(r\sin\theta + \theta)$$ **Ans.**

Example *If* $u = (1 - 2xy + y^2)^{-1/2}$ *prove that,*

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3.$$

Solution. $u = (1 - 2xy + y^2)^{-1/2}$ ...(1)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y)$$

$$x\frac{\partial u}{\partial x} = xy(1 - 2xy + y^2)^{-3/2} \quad ...(2)$$

Differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y)$$

$$y\frac{\partial u}{\partial y} = (xy - y^2)(1 - 2xy + y^2)^{-3/2} \quad ...(3)$$

Subtracting (3) from (2), we get

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy\,(1 - 2xy + y^2)^{-3/2} - (xy - y^2)\,(1 - 2xy + y^2)^{-3/2}$$

$$= y^2\,(1 - 2xy + y^2)^{-3/2}$$

$$= y^2u^3.$$

Proved.

Practice Questions

find all the 1st order partial derivatives.

1. $f(x, y, z) = 4x^3y^2 - \mathbf{e}^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$

2. $w = \cos\left(x^2 + 2y\right) - \mathbf{e}^{4x - z^4 y} + y^3$

3. $f(u, v, p, t) = 8u^2t^3p - \sqrt{v}\,p^2t^{-5} + 2u^2t + 3p^4 - v$

4. $f(u, v) = u^2\sin\left(u + v^3\right) - \sec(4u)\tan^{-1}(2v)$

ANSWERS

1. $$\frac{\partial f}{\partial x} = f_x = 12x^2y^2 - \frac{2z^3}{x^3} - 16x^{15}$$
$$\frac{\partial f}{\partial y} = f_y = 8x^3y - 4\mathbf{e}^z y^3 + 4$$
$$\frac{\partial f}{\partial z} = f_z = -\mathbf{e}^z y^4 + \frac{3z^2}{x^2}$$

2. $$\frac{\partial w}{\partial x} = w_x = -2x\sin\left(x^2 + 2y\right) - 4\mathbf{e}^{4x - z^4 y}$$
$$\frac{\partial w}{\partial y} = w_y = -2\sin\left(x^2 + 2y\right) + z^4\mathbf{e}^{4x - z^4 y} + 3y^2$$
$$\frac{\partial w}{\partial z} = w_z = 4z^3 y\mathbf{e}^{4x - z^4 y}$$

3. $$\frac{\partial f}{\partial u} = f_u = 16ut^3p + 4ut$$
$$\frac{\partial f}{\partial v} = f_v = -\frac{1}{2}v^{-\frac{1}{2}}p^2t^{-5} - 1$$
$$\frac{\partial f}{\partial p} = f_p = 8u^2t^3 - 2\sqrt{v}\,pt^{-5} + 12p^3$$
$$\frac{\partial f}{\partial t} = f_t = 24u^2t^2p + 5\sqrt{v}\,p^2t^{-6} + 2u^2$$

4. $$\frac{\partial f}{\partial u} = f_u = 2u\sin\left(u + v^3\right) + u^2\cos\left(u + v^3\right)$$
$$- 4\sec(4u)tan\,(4u)\tan^{-1}(2v)$$
$$\frac{\partial f}{\partial v} = f_v = 3v^2u^2cos\,\left(u + v^3\right) - \frac{2\sec(4u)}{1 + 4v^2}$$

LECTURE-17

Problems Based on Partial Derivatives

Example 1. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ where $u(x, y) = \log_e \left(\frac{x^2 + y^2}{xy}\right)$

Sol. We have $$u(x, y) = \log_e \left(\frac{x^2 + y^2}{xy}\right)$$

$$\Rightarrow \quad u(x, y) = \log (x^2 + y^2) - \log x - \log y \quad \text{...}(i)$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

Now differentiating partially w.r.t. y.

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{4xy}{\left(x^2 + y^2\right)^2} \quad \text{...}(A)$$

Again differentiate (i) partially w.r.t. y, we obtain

$$\frac{\partial u}{\partial y} = \frac{2y}{\left(x^2 + y^2\right)} - \frac{1}{y}$$

Next, we differentiate above equation w.r.t. x.

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{\left(x^2 + y^2\right)^2} \quad \text{...}(B)$$

Thus, from (A) and (B), we find

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$ **Hence proved.**

Example 2. If $f = \tan^{-1}\left(\frac{y}{x}\right)$, verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Sol. We have $$f = \tan^{-1}\left(\frac{y}{x}\right) \quad ...(i)$$

Differentiating (*i*) partially with respect to *x*, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \left(\frac{-y}{x^2+y^2}\right) \quad ...(ii)$$

Differentiating (*i*) partially with respect to *y*, we get

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2} \quad ...(iii)$$

Differentiating (*ii*) partially with respect to *y*, we get

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right)(-1)-(-y)(2y)}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad \ldots(iv)$$

Differentiating (*iii*) partially with respect to *x*, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{\left(x^2 + y^2\right)(1) - x(2x)}{\left(x^3 + y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad \ldots(v)$$

∴ From eqns. (*iv*) and (*v*), we get $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. **Hence proved.**

Example 3. If $f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ then prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Sol.

$$\frac{\partial f}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right)$$

or

$$\frac{\partial f}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating both sides with respect to y, we get

$$\frac{\partial^2 f}{\partial y \partial x} = 2x \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2} \qquad ...(i)$$

Again

$$\frac{\partial f}{\partial y} = x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right)$$

or

$$\frac{\partial f}{\partial y} = \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) = x - 2y \tan^{-1}\left(\frac{x}{y}\right).$$

Differentiating both sides with respect to x, we get

$$\frac{\partial^2 f}{\partial x\, \partial y} = 1 - 2y \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$ **Hence proved.**

Example 4. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$; show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Sol $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

so we have to find partial derivative of u w.r.t. x, y, z

Sol. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \quad \ldots(i)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \quad \ldots(ii)$$

and

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}. \quad \ldots(iii)$$

Adding eqns. (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

| As $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

or $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}. \quad \text{...}(iv)$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ from } (iv)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}.$$

Hence proved.

Example 5. If $z = f(x - by) + \phi(x + by)$, prove that

$$b^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}.$$

Sol. Given $\quad z = f(x - by) + \phi(x + by) \quad \ldots(i)$

$\therefore \quad \frac{\partial z}{\partial x} = f'(x - by) + \phi'(x + by)$

and $\quad \frac{\partial^2 z}{\partial x^2} = f''(x - by) + \phi''(x + by). \quad \ldots(ii)$

Again from (i), $\quad \frac{\partial z}{\partial y} = -bf'(x - by) + b\phi'(x + by)$

and $\quad \frac{\partial^2 z}{\partial y^2} = b^2 f''(x - by) + b^2\phi''(x + by) = b^2 \frac{\partial^2 z}{\partial x^2}$, from (ii).

Q.5 If $x^2 = au + bv$, $y^2 = au - bv$, prove that

(i) $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2}$ (2017-18)

(ii) $\left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{2}$

Sol. Given $x^2 = au + bv$ — ① and $y^2 = au - bv$ — ②

Adding ① + ②, $x^2 + y^2 = 2au \Rightarrow u = \frac{1}{2a}(x^2 + y^2)$ — ③

Subtracting ② from ①, $x^2 - y^2 = 2bv \Rightarrow v = \frac{1}{2b}(x^2 - y^2)$ — ④

(i) Diff. eq ① partially w.r.t. u,

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a.1 + 0 \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

Now diff. eq ③ partially w.r.t. x, $\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \cdot \frac{a}{2x} = \frac{1}{2}$$

(ii) Diff. eq ② partially w.r.t. v,

$$2y\left(\frac{\partial y}{\partial v}\right)_u = 0 - b.1 \Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$$

Now diff. eq ④ partially w.r.t. y, $\left(\frac{\partial v}{\partial y}\right)_x = \frac{1}{2b}(-2y) = -\frac{y}{b}$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{b}{2y}\right)\left(-\frac{y}{b}\right) = \frac{1}{2}$$

Q. If $u = f(r)$, where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$ (2015-16)

Sol. $r^2 = x^2 + y^2$; diff. partially w.r.t. x & y,

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now $u = f(r)$

$$\therefore \frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) + x\cdot\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) f'(r) + \frac{x}{r} f''(r)\cdot\frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw) = vw\frac{\partial u}{\partial x} + uw\frac{\partial v}{\partial x} + uv\frac{\partial w}{\partial x}\right]$$

$$= \frac{1}{r} f'(r) - \frac{x}{r^2}\cdot\frac{x}{r} f'(r) + \frac{x}{r}\cdot\frac{x}{r} f''(r)$$

$$= \left(\frac{1}{r} - \frac{x^2}{r^3}\right) f'(r) + \frac{x^2}{r^2} f''(r)$$

$$= \frac{r^2 - x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \qquad (\because r^2 = x^2 + y^2)$$

$$= \frac{y^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r)$$

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r)$

($\because$ u is symmetric about x & y, $\therefore$ replace x by y to find $\frac{\partial^2 u}{\partial y^2}$)

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2}{r^3} f'(r) + \frac{x^2+y^2}{r^2} f''(r)$$

$$= \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} f''(r) = f''(r) + \frac{1}{r} f'(r)$$

Ans.

Practice Questions

1. If $V = \left(x^2 + y^2 + z^2\right)^{-1/2}$, prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

2. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

3. If $z = f(x + ct) + \varphi(x - ct)$, show that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

4. If $w = \sqrt{x^2 + y^2 + z^2}$ & $x = \cos v, y = u \sin v, z = uv$, then prove that $\left[u \frac{\partial w}{\partial u} - v \frac{\partial w}{\partial v}\right] = \frac{u}{\sqrt{1+v^2}}$. (Long question) 2016-17

LECTURE-18

Total Derivative

Total Derivative

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$, which is called the total derivative of u.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Cor. 1. If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is called a composite function of (two variables) u and v and we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Cor. 2. If $u = f(x, y, z)$ and x, y, z are functions of t then total derivative of u is $\frac{du}{dt}$, given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Q1. Find $\frac{du}{dt}$ if $u = x^3 + y^3$, $x = a\cos t$, $y = b\sin t$

(2019-20)

Sol

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= 3x^2 \cdot (-a\sin t) + 3y^2 (b\cos t)$$

$$= -3(a\cos t)^2 a\sin t + 3(b\sin t)^2 b\cos t$$

$$= 3\sin t\cos t\,[b^3\sin t - a^3\cos t]$$

Q2. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.

(2014-15)

Sol. $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$

$= 2x \cdot 2e^{2t} + 2y\cdot(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2z(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= 4e^{2t}\cdot e^{2t} + 2e^{2t}\cos 3t\,(2e^{2t}\cos 3t - 3e^{2t}\sin 3t)$
$+ 2e^{2t}\sin 3t\,(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t\cos 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t\cos 3t$

$= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t)$

$= 4e^{4t} + 4e^{4t} = 8e^{4t}$

Verification (by direct substitution)

Put values of x, y and z in given u, we get

$u = (e^{2t})^2 + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t$

$= e^{4t} + e^{4t} = 2e^{4t}$

$\therefore \frac{du}{dt} = 2 \cdot 4e^{4t} = 8e^{4t}.$

Q3. If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

(2020-21)

Sol: Let $X = y-z$, $Y = z-x$, $Z = x-y$

then $u = f(X, Y, Z)$ is a composite function of x, y, z

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}+\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}=0$$

Q.4. If $u=f(2x-3y,\ 3y-4z,\ 4z-2x)$, prove that

$$\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0$$

(2019-20)

Sol Let $X=2x-3y$, $Y=3y-4z$, $Z=4z-2x$

$\therefore$ u is a composite function of x, y and z.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}=\frac{\partial u}{\partial X}(2)+\frac{\partial u}{\partial Y}(0)+\frac{\partial u}{\partial Z}(-2)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}=\frac{\partial u}{\partial X}(-3)+\frac{\partial u}{\partial Y}(3)+\frac{\partial u}{\partial Z}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial x}(0) + \frac{\partial u}{\partial y}(-4) + \frac{\partial u}{\partial Z}(4)$$

$$L.H.S. = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial x} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial Z} = 0 = R.H.S.$$

Q.5 If $V = f(2x-3y,\ 3y-4z,\ 4z-2x)$, Prove that (2014-15)

$$6\frac{\partial V}{\partial x} + 4\frac{\partial V}{\partial y} + 3\frac{\partial V}{\partial z} = 0$$

Sol Proceed as above Q.

Q6 If $u = f(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ (2017-18)

Sol Proceed as above

Hint: Here u is composite function of r, s, t

Cor 3. If $u = f(x, y)$ where $y = \phi(x)$, then u is a composite function of x.

$$\therefore \boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}}$$

Q.1. If $u = x \log(xy)$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$

Sol. $\frac{du}{dx}$ shows that u is function of single variable x, so taking y as a function of x, use the formula

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \text{--- (1)}$$

Now diff. u, partially w.r.t. x & y respectively.

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\cdot y\right) + \log xy = 1 + \log xy$$

and $$\frac{\partial u}{\partial y} = x\cdot\left(\frac{1}{xy}\cdot x\right) = \frac{x}{y}$$

Also differentiating $x^3 + y^3 + 3xy = 1$, w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x^2+y}{y^2+x}\right)$$

Put all these values in eq. ①,

$$\frac{du}{dx} = 1 + \log xy - \frac{x(x^2+y)}{y(y^2+x)}$$ Ans.

Implicit Function – A function $f(x, y) = c$ is called implicit, if it can't be expressible as $x = \phi(y)$ or $y = \phi(x)$.

For Ex. $x^3 + y^3 + 3xy = 1$, $x^y + y^x = xy$ etc.

Cor. 4. If $f(x, y) = c$ is an implicit function

then $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}$$

Q.2. If $f(x,y) = 0$, $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Sol.

$f(x,y) = 0$ gives $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$\phi(y, z) = 0$ gives $\dfrac{dz}{dy} = -\dfrac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$

Now $\dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx} \Rightarrow \dfrac{dz}{dx} = \dfrac{\frac{\partial \phi}{\partial y} \quad \frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial z} \quad \frac{\partial f}{\partial y}}$

$\Rightarrow \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial \phi}{\partial z} \cdot \dfrac{dz}{dx} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial \phi}{\partial y}$

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Q3. If $f(x,y,z,w)=0$, then find $\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$

(2015-16)
(2017-18)

Sol.

$$\frac{\partial x}{\partial y} = -\frac{f_y}{f_x}, \quad \frac{\partial y}{\partial z} = -\frac{f_z}{f_y}$$

$$\frac{\partial z}{\partial w} = -\frac{f_w}{f_z}, \quad \frac{\partial w}{\partial x} = -\frac{f_x}{f_w}$$

Now $$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x} = \left(-\frac{f_y}{f_x}\right)\left(-\frac{f_z}{f_y}\right)\left(-\frac{f_w}{f_z}\right)\left(-\frac{f_x}{f_w}\right)$$

$$= 1.$$

Example . *If $z = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$, show that*

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}.$$

(M.U. 2009; Nagpur Univesity 2002)

Solution. We have,

$$x = e^u \cos v, \qquad y = e^u \sin v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} e^u \cos v + \frac{\partial z}{\partial y} e^u \sin v = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$y\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \qquad \text{...(1)}$$

And

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v) = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial v} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \qquad ...(2)$$

On adding (1) and (2), we get

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = (x^2 + y^2)\frac{\partial z}{\partial y} = (e^{2u}\cos^2 v + e^{2u}\sin^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}(\cos^2 v + \sin^2 v)\frac{\partial z}{\partial y} = e^{2u}\frac{\partial z}{\partial y}$$

Proved.

Example . *A function f (x, y) is rewritten in terms of new variables*

$$u = e^x \cos y, \qquad v = e^x \sin y$$

Show that (i) $\dfrac{\partial f}{\partial x} = u\dfrac{\partial f}{\partial u} + v\dfrac{\partial f}{\partial v}$ *and* (ii) $\dfrac{\partial f}{\partial y} = -v\dfrac{\partial f}{\partial u} + u\dfrac{\partial f}{\partial v}$

Solution. $u = e^x \cos y, \quad \dfrac{\partial u}{\partial x} = e^x \cos y = u, \quad \Rightarrow \quad \dfrac{\partial u}{\partial y} = -\, e^x \sin y = -\, v$

$$v = e^x \sin y, \quad \frac{\partial v}{\partial x} = e^x \sin y = v, \quad \frac{\partial v}{\partial y} = e^x \cos y = u$$

(*i*) We know that $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}u + \frac{\partial f}{\partial v}v \qquad \text{... (1)} \quad \textbf{Proved.}$$

(*ii*) $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}.(-v) + \frac{\partial f}{\partial v}u = -v.\frac{\partial f}{\partial u} + u.\frac{\partial f}{\partial v} \qquad \text{... (2)} \quad \textbf{Proved.}$$

Practice Questions

1. If $V = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial V}{\partial x} + 4\frac{\partial V}{\partial y} + 3\frac{\partial V}{\partial z} = 0.$

2. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t.$

Ans. $2e^{4t}$

3. If $x = e^r \cos\theta, y = e^r \sin\theta$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right).$

4. If $z = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$; show that $\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}.$

LECTURE-19

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Euler's Theorem for Homogeneous Functions

Homogeneous function

A function f(x,y) is said to be homogeneous of degree n in variable x and y if it can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ OR $y^n\phi\left(\frac{x}{y}\right)$.

An alternative test

A function f(x,y) will be homogeneous of degree n if $f(tx,ty)=t^n f(x,y)$.

Example: $f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Replacing x→tx, y→ty $\quad f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})}$

$f(tx,ty)=t^{1/2}f(x,y)$

So f(x,y) is homogeneous function of degree ½.

Similarly a function f(x,y,z) is said to be homogeneous of degree n in the variables x, y and z if $f(tx,ty,tz)=t^n f(x,y,z)$

Euler's Theorem on Homogeneous Functions.

If u is a homogeneous function of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$ where $u \equiv u(x, y)$

Note – ① If u is a homogeneous function of degree n in x, y and z then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

② If u is a homogeneous function of degree n in x & y then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

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Q.1. If $u = x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ (2018-19)

Sol. $u(tx, ty) = (tx)^3 (ty)^2 \sin^{-1}\left(\frac{ty}{tx}\right) = t^5 x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$= u(x, y)$

$\therefore$ u is homogeneous function of order $n = 5$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 5u$$

Q8. If $V = (x^2+y^2+z^2)^{-\frac{1}{2}}$ then find $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$

Sol (2015-16)

$$V(tx, ty, tz) = [(tx)^2+(ty)^2+(tz)^2]^{-\frac{1}{2}}$$

$$= t^{-1}[x^2+y^2+z^2]^{-\frac{1}{2}} = t^{-1}V(x,y,z)$$

∴ V is homogeneous function of order $n=-1$

By Euler's Theorem,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = nV = -V$$

Q3. If $u(x,y) = (\sqrt{x} + \sqrt{y})^5$, find the value of $\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right)$ (2012-13)

Sol $u(tx, ty) = (\sqrt{tx} + \sqrt{ty})^5 = t^{\frac{5}{2}} (\sqrt{x} + \sqrt{y})^5 = t^{\frac{5}{2}} u(x,y)$

$\therefore$ u is homogeneous function of order $n = \frac{5}{2}$.

By Euler's theorem

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u = \frac{5}{2}\left(\frac{5}{2} - 1\right)u = \frac{5}{2} \cdot \frac{3}{2} u = \frac{15}{4} u.$$

Q4. If $u = x^2yz - 4y^2z^2 + 2xz^3$, then find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}.$$

(2011-12)

Sol. Proceed as above. (Hint: order $n = 4$)

Ques. If $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$, find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

Sol. Let $u(x,y) = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right)$

and $v(x,y) = \log x - \log y = \log \frac{x}{y}$

Now $u(tx, ty) = t^6 x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) = t^6 u(x,y)$

$\therefore$ u is a homogeneous function of degree 6.

Also $v(tx, ty) = \log\left(\frac{tx}{ty}\right) = \log\left(\frac{x}{y}\right) = v(x, y)$

$\therefore v$ is a homogeneous function of degree 0.

By Euler's theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u$; and $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0$

Adding them, we get

$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 6u$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$ $(\because z = u + v)$

1/3 1/3

Q.6. Verify Euler's theorem for the function $z = \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

Sol $z(tx, ty) = \dfrac{(tx)^{1/3} + (ty)^{1/3}}{(tx)^{1/2} + (ty)^{1/2}} = \dfrac{t^{\frac{1}{3}}}{t^{\frac{1}{2}}} \dfrac{(x^{1/3} + y^{1/3})}{(x^{1/2} + y^{1/2})} = t^{-1/6} z(x, y)$

$\therefore z(x, y)$ is homogeneous function of order $n = -\frac{1}{6}$

By Euler's Theorem $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -\dfrac{1}{6}u$ [2015-16, 2017-18]

Verification -

$$\frac{\partial z}{\partial x} = \frac{(x^{1/2} + y^{1/2})(\frac{1}{3} x^{-2/3}) - (x^{1/3} + y^{1/3})(\frac{1}{2} x^{-1/2})}{(x^{1/2} + y^{1/2})^2}$$

$$x\frac{\partial z}{\partial x} = \frac{1}{(x^{1/2}+y^{1/2})^2}\left[\frac{1}{3}\left(x^{5/6}+x^{\frac{1}{3}}y^{\frac{1}{2}}\right)-\frac{1}{2}\left(x^{\frac{5}{6}}+x^{\frac{1}{2}}y^{\frac{1}{3}}\right)\right]$$

$$\frac{\partial z}{\partial y} = \frac{(x^{1/2}+y^{1/2})\left(\frac{1}{3}y^{-2/3}\right)-(x^{1/3}+y^{1/3})\left(\frac{1}{2}y^{-1/2}\right)}{(x^{1/2}+y^{1/2})^2}$$

$$y\frac{\partial z}{\partial y} = \frac{1}{(x^{1/2}+y^{1/2})^2}\left[\frac{1}{3}\left(x^{1/2}y^{1/3}+y^{5/6}\right)-\frac{1}{2}\left(x^{1/3}y^{1/2}+y^{5/6}\right)\right]$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \frac{(-1/6)}{(x^{1/2}+y^{1/2})^2}\left[x^{5/6}+y^{5/6}+x^{\frac{1}{3}}y^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{3}}\right]$$

$$= -\frac{1}{6}\frac{1}{(x^{1/2}+y^{1/2})^2}\left[x^{\frac{1}{2}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)+y^{\frac{1}{2}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\right] = -\frac{1}{6}\frac{\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)}{(x^{1/2}+y^{1/2})}$$

$$= -\frac{1}{6}u$$

Hence verified

Q. *Verify Euler's theorem for the functions :*

$$f(x, y) = ax^2 + 2hxy + by^2$$

Sol. (i) $f(x, y) = ax^2 + 2hxy + by^2$

$$= x^2\left(a + 2h\frac{y}{x} + \frac{by^2}{x^2}\right)$$

$$= x^2\,\phi(y/x), \text{ where } \phi\left(\frac{y}{x}\right)$$

$$= a + 2h\,\frac{y}{x} + \frac{by^2}{x^2}$$

i.e., $f(x, y)$ is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem, we should have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

Now $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(ax^2 + 2hxy + by^2) = 2ax + 2hy$

$$\Rightarrow \quad x\frac{\partial f}{\partial x} = 2ax^2 + 2hxy \qquad \text{...(1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(ax^2 + 2hxy + by^2) = 2hx + 2by$$

$$\Rightarrow \quad y\frac{\partial f}{\partial y} = 2hxy + 2by^2 \qquad \text{...(2)}$$

Adding (1) and (2), we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2ax^2 + 4hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2) = 2f$$

Hence Euler's theorem is verified.

Example (*i*) *If* $u = \cos\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

Sol. (*i*) $$u(x, y, z) = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$$

$\therefore$ $$u(tx, ty, tz) = \cos\left\{\frac{t^2(xy + yz + zx)}{t^2(x^2 + y^2 + z^2)}\right\}$$

$$= t^\circ \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right) = t^\circ\, u(x, y, z)$$

Hence u is a homogeneous function in x, y and z of degree 0.

Hence by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0\,.\,u = 0$$

Q. *If* $u = (x^2 + y^2)^{1/3}$, *show that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}.$$

Sol. Given $u = (x^2 + y^2)^{1/3} = x^{2/3}\left(1 + \frac{y^2}{x^2}\right)^{1/3}$

$$= x^{2/3} \phi\left(\frac{y}{x}\right), \text{ say,}$$

where $\phi\left(\frac{y}{x}\right) = \left(1 + \frac{y^2}{x^2}\right)^{1/3}$

i.e., u is a homogeneous function of degree $\frac{2}{3}$, in x, y.

$\therefore$ By Euler's Theorem of higher order,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= n(n-1)\, u = \frac{2}{3}\left(\frac{2}{3} - 1\right) u = -\frac{2}{9} u.$$

Practice Questions

1. If $z = xy/(x+y)$, find the value of $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2}$. **Ans. 0**

2. If $v = \dfrac{x^3y^3}{x^3+y^3}$, show that $x.\dfrac{\partial v}{\partial x} + y.\dfrac{\partial v}{\partial y} = 3v$.

3. **Prove that $g(x, y) = x \log (y/x)$ is homogeneous Verify Euler's Theorem for g.**

4. If $u(x,y) = \dfrac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2}u$.

LECTURE 20

Deductions from Euler's Theorem

Deductions from Euler's Theorem

If $F(u) = f(x, y)$, where $f(x, y)$ is a homogeneous function in x and y of degree n, then

(i) $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{F(u)}{F'(u)}$$

(ii) $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u) - 1]$$

Where $\phi(u) = n \dfrac{F(u)}{F'(u)}$

Note- If $F(u) = f(x, y, z)$, where $f(x, y, z)$ is a homogeneous function in x, y and z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{F(u)}{F'(u)}$$

Q1. If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u.$$ Also evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

(2020-21)

Sol. Here u is not a homogeneous function. But

$\sec u = \frac{x^3 - y^3}{x+y}$ is of the form $F(u) = f(x,y)$

$$f(tx, ty) = \frac{(tx)^3 - (ty)^3}{tx + ty} = t^2\,\frac{x^3 - y^3}{x+y} = t^2 f(x,y)$$

$x+y$

$\therefore f(x,y)$ is a homogeneous function of order $n=2$

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{F(u)}{F'(u)}=2\frac{\sec u}{\sec u.\tan u}=2.\cot u$$

Now $\phi(u)=2\cot u$

By deduction from Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)[\phi'(u)-1]$$

$$=2\cot u\,[-2\operatorname{cosec}^2 u-1]$$

$$=-2\cot u\,[2\operatorname{cosec}^2 u+1]$$

Q2. If $u=\sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u$$ (2017-18)

Sol: u is not a homogeneous function. But

$\sin u=\dfrac{x^3+y^3+z^3}{ax+by+cz}$, which is of the form $F(u)=f(x,y,z)$

$$f(tx,ty,tz)=\frac{(tx)^3+(ty)^3+(tz)^3}{atx+bty+ctz}=t^2\,\frac{x^3+y^3+z^3}{ax+by+cz}=t^2 f(x,y,z)$$

Hence $f(x, y, z)$ is a homogeneous function of order 2.

By deduction of Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{F(u)}{F'(u)} = 2\frac{\sin u}{\cos u} = 2\tan u.$$

Q3. If $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$, prove that

(i) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2 \sin u . \cos 3u$

Sol. Here u is not a homogeneous function. But

$$F(u) = \tan u = \frac{x^3 + y^3}{x - y} = f(x, y)$$

is a homogeneous function of order 2.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{F(u)}{F'(u)} = \frac{2 \tan u}{\sec^2 u} = 2 \frac{\sin u}{\cos u} . \cos^2 u$$

$$= 2 \sin u \cos u$$

$$= \sin 2u$$

(ii) Here $\phi(u) = \sin 2u$, $\therefore \phi'(u) = 2\cos 2u$

By deduction from Euler's theorem,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \phi(u)\,[\phi'(u) - 1]$$

$$= \sin 2u\,[2\cos 2u - 1] = \sin 4u - \sin 2u$$

$$= 2\cos 3u \cdot \sin u .$$

Example 4 : If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u=0$$

Solution : Here given $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$

$$\Rightarrow \sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} = f \text{ (say)}$$

Now here f is a homogeneous function in x, y, z of degree (1 – 4) i.e –3.
Hence, by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = -3f$$

$$\text{or } x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = -3 \sin u$$

$\therefore f = \sin u$

$$\text{or } x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} = -3\sin u$$

$$\text{or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u}$$

$$\text{or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0.$$

Hence Proved

Example 5 *If* $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, *show that:*

(*i*) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\,\phi\left(\frac{y}{x}\right)$ (*ii*) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$

Sol. Let $v = x\,\phi\left(\frac{y}{x}\right)$ and $w = \psi\left(\frac{y}{x}\right) = x^0\,\psi\left(\frac{y}{x}\right)$ so that,

$$u = v + w \quad \text{...(1)}$$

(*i*) Since v is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v \quad \text{...(2)}$$

Since w is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0 \quad \text{...(3)}$$

Adding equations (2) and (3), we get

$$x\,\frac{\partial}{\partial x}(v + w) + y\,\frac{\partial}{\partial y}(v + w) = v$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\,\phi\left(\frac{y}{x}\right).$$

(*ii*) Since v is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)\,v = 1\,(1-1)\,v = 0 \quad \text{...(4)}$$

Also since w is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = n(n-1)\,w = 0\,(0-1)\,w = 0 \quad \text{...(5)}$$

Adding (4) and (5), we get

$$x^2 \frac{\partial^2}{\partial x^2}(v+w) + 2xy \frac{\partial^2}{\partial x \partial y}(v+w) + y^2 \frac{\partial^2}{\partial y^2}(v+w) = 0$$

$$\Rightarrow \qquad x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = 0.$$

Practice Questions

1. If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0.$

2. If $u = \sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\tan u.$

3. If $u = \sin^{-1}\left(\dfrac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$, then evaluate the value of $\left(x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial u}{\partial y}\right)$

4 If $u = \sec^{-1}\left(\dfrac{x^3-y^3}{x+y}\right)$, show that $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = 2\cot u$, then evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

Ans. $-2\cot u\,(2\operatorname{cosec}^2 u + 1)$.

Practice Questions

Q5. Show that $xU_x + yU_y + zU_z = -2\cot u$, where

$$u = \cos^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$$

(2013-14)

Q6. If $u = \sin^{-1}\left(\frac{x^{1/3} + y^{1/3}}{x^{1/2} - y^{1/2}}\right)^{1/2}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u.$$

(2011-12)

Q7. Prove that $xu_x + yu_y = \frac{5}{2}\tan u$ if $u = \sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$

(2014-15)

Ques. If $u = \frac{x^2y^2}{x^2+y^2} + \cos\left(\frac{xy}{x^2+y^2}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\frac{x^2y^2}{x^2+y^2}.$$

2022-23

Answers

(3) $\tan u(\sec^2 u-12)/144$

Lecture 44

Curve Tracing

Introduction

The knowledge of curve tracing is to avoid the labour of plotting a large number of points. It is helpful in finding the length of curve, area volume and surface area. The limits of integration can be easily found on tracing the curve.

This method is used in Cartesian Equation.

1. **Symmetry:**

   (a) A curve is symmetric about x-axis if the equation remains the same by replacing y by –y. here y should have even powers only.

   For ex: $y^2=4ax$.

   (b) It is symmetric about y-axis if it contains only even powers of x.

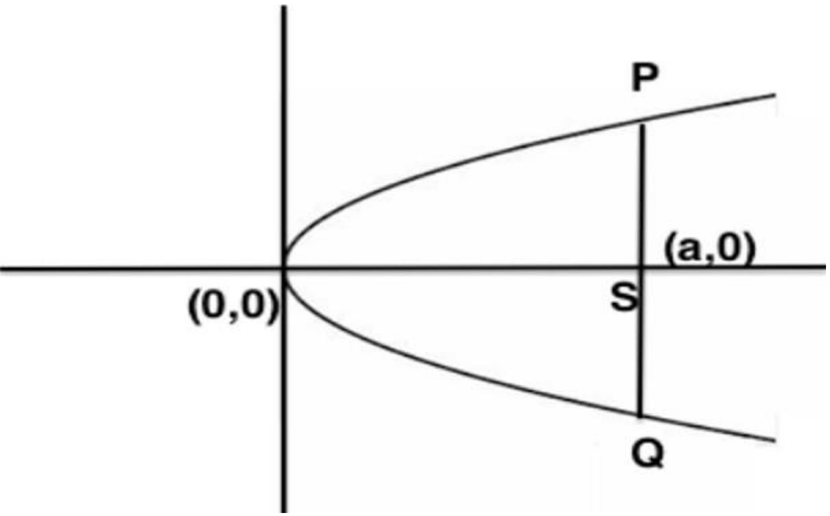
   For ex: $x^2=4ay$

   (c) If on interchanging x and y, the equation remains the same then the curve is symmetric about the line y=x.

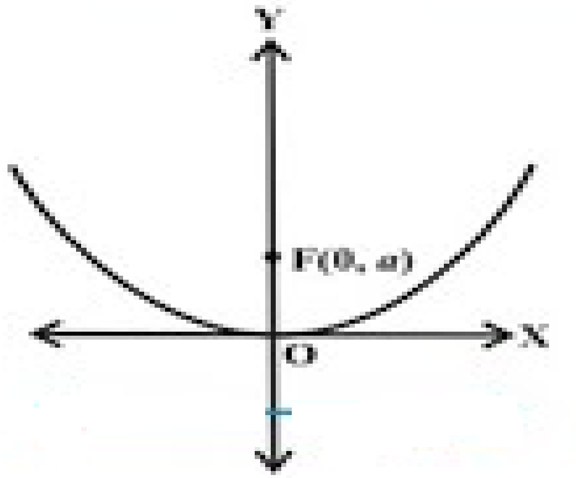
   For ex: $x^3+y^3=3axy$

   (d) A curve is symmetric in the opposite quadrants if its equation remains the same where x and y replaced by –x and –y respectively.

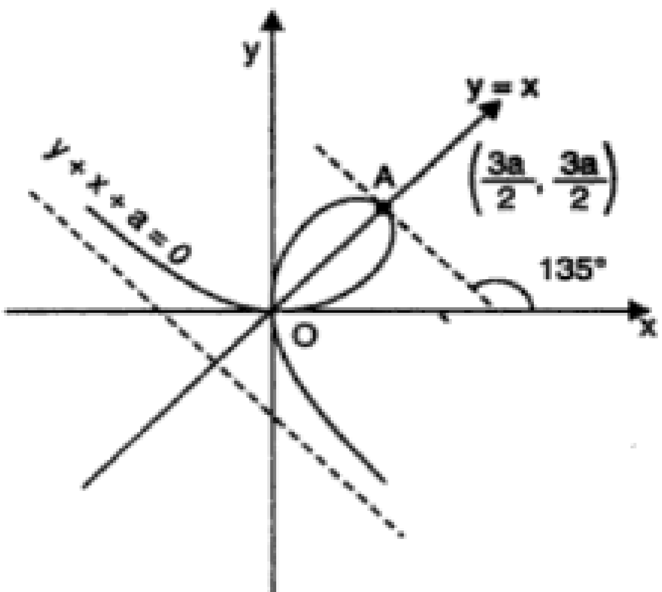
   For ex: $xy=c$



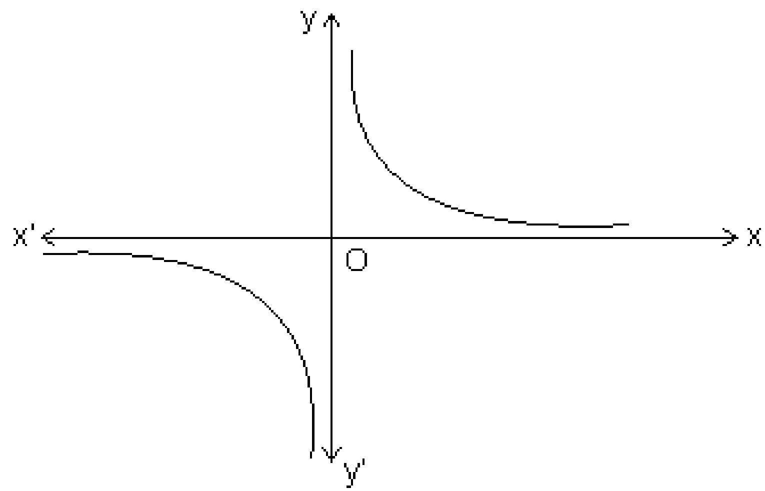
Symmetry about x – axis



Symmetry about y – axis



Symmetry about y = x



Symmetry in the opposite quadrant

2. (a) **Curve through origin:**

The curve passes through the origin, if the equation does not contain constant term.

For ex: the curve $y^2=4ax$ passes through the origin.

(b) **Tangent at the origin:**

To know the nature of a multiple point it is necessary to find the tangent at that point.

The equation of the tangent at the origin can be obtained by equating to zero, the lowest degree term in the equation of the curve.

3. **The points of intersection with the axes:**

(a) By putting y=0 in the equation of the curve we get the co-ordinates of the point of intersection with the x-axis.

For ex: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ put y=0 we get $x=\pm a$

Thus, (a, 0) and (-a, 0) are the co-ordinates of point of intersection.

(b) By putting x=0 in the equation of the curve, the co-ordinate of the point of intersection with the y-axis is obtained by solving the new equation.

4. **Regions in which the curve does not lie:**

If the value of y is imaginary for certain value of x then the curve does not exist for such values.

Example 1: $y^2 = 4x$

Answer: For negative value of x, if y is imaginary then there is no curve in second and third quadrants.

Example 2: $a^2x^2 = y^3(2a - y)$.

Answer: (i) For y>2a, x is imaginary. There is no curve beyond y=2a.

(ii) For negative value of y, if x is imaginary then there is no curve in 3rd and 4th quadrants.

5. **Asymptotes are the tangents to the curve at infinity:**

(a)**Asymptote parallel to the x-axis** is obtained by equating to zero, the coefficient of the highest power of x.

For ex: $yx^2 - 4x^2 + x + 2 = 0$

⇨ $(y - 4)x^2 + x + 2 = 0$

The coefficient of the highest power of x $i.e\ x^2$ is $y-4=0$

$\therefore\ y-4=0$ is the asymptote parallel to the x-axis.

(b) **Asymptote parallel to the y-axis** is obtained by equating to zero, the coefficient of highest power of y.

For ex: $xy^3-2y^3+y^2+x^2+2=0$

⇨ $(x-2)y^3+y^2+x^2+2=0$

The coefficient of the highest power of y $i.e.$ y^3 is $x-2$.

$\therefore x-2=0$ is the asymptote parallel to y-axis.

(c) Oblique asymptotes

(i) Find $\phi_n(m)$ by putting $x=1$ and $y=m$ in the highest degree terms of the curve

(ii) Solve $\phi_n(m)=0$ for m.

(iii) Find $\phi_{n-1}(m)$ by putting $x=1$ and $y=m$ in the next highest $(n-1)$ degree terms of the curve

(iv) Find c by $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$

If value of m is repeated two times then find c by $\frac{c^2}{\lfloor 2}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$

(V) Obtain the equation of asymptotes by putting the values of m and c in $y = mx + c$.

Example:- Find the oblique asymptote of
$xy^2 = 4a^2(2a - x)$, if it has.
[2013, 2015]

Solution:- $xy^2 = 4a^2(2a - x)$

$\phi_3(m) = m^2 = 0$ we get $m = 0, 0$

$\phi_2(m) = 0$ and $\phi_3'(m) = 2m$, $\phi_3''(m) = 2$

$\phi_1(m) = 4a^2$

use formula

$$\frac{c^2}{\lfloor 2}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2} \cdot 2 + 4a^2 = 0$$

$c^2 = -4a^2$ (c becomes imaginary)

curve has no real asymptote (oblique).

Note: but the curve has one real asymptote which is parallel to y-axis which is

$$x = 0$$

6. Tangent:- Put $\frac{dy}{dx} = 0$ for the points where tangent is parallel to x-axis

e.g. $x^2 + y^2 - 4x + 4y - 1 = 0$ gives $\frac{dy}{dx} = \frac{4 - 2x}{2y + 4}$

$\frac{dy}{dx} = 0$ gives $x = 2$,

for solving y the curve when $x = 2$ we get

$$y = 1, -5$$

at $(2, 1)$ & $(2, -5)$ tangents are parallel to x-axis

7. Table :- Prepare a table for certain values of x & y

e.g. $y^2 = 4x + 4$

x	-1	0	1	2	3
y	0	± 2	$\pm 2\sqrt{2}$	$\pm 2\sqrt{3}$	± 4

Remember **POSTER**

P = point of intersection

O = Origin

S = Symmetry

T = Tangent

A = Asymptote

R = Region

IMPORTANT DEFINITIONS:

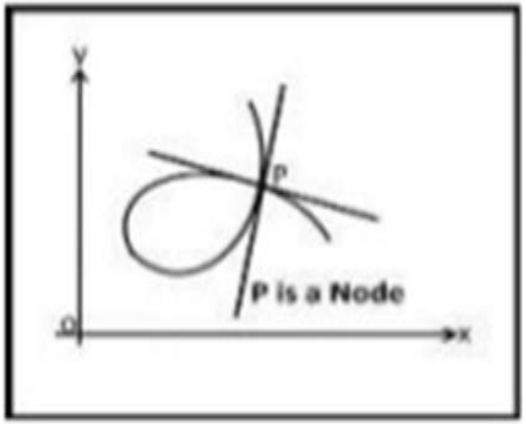
(I) **Singular Points:** This is an unusual point on a curve.

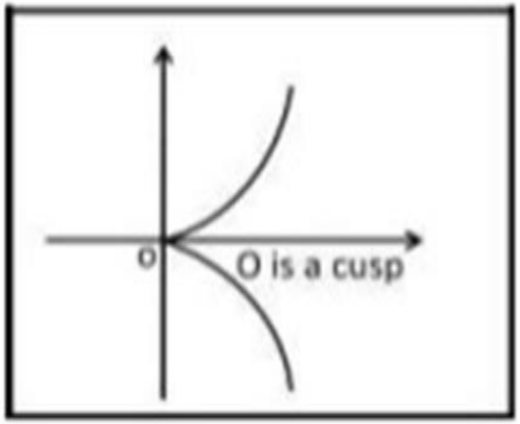
(II) **Multiple points:** A point through which a curve passes more than one time.

(III) **A double Point:** If a curve passes two times through a point, then this point is called a double point.

(a) **Node:** A double point at which two real tangents (not coincident) can be drawn.

(b) **Cusp:** A double point is called cusp if the two tangents at it are coincident.





(IV) **Point of inflexion: A point** where the curve crosses the tangent is called a point of inflexion.

(V) **Conjugate point:** This is an isolated point. In its neighbour there is no real point of the curve.

At each double point of the curve y=f(x), we get,

$$D = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2}$$

a) If D is +ve, double point is a node or conjugate point.
b) If D is 0, double point is a cusp or conjugate point.
c) If D is −ve, double point is a conjugate point.

Example:- Discuss all symmetries of the curve

$$x^2y^2 = x^2 - a^2$$

Solution :- We have $x^2y^2 = x^2 - a^2$

(i) Since the curve has even powers of x, so the curve is symmetric about y-axis.

(ii) Since the curve has even powers of y, so the curve is symmetric about x-axis.

(iii) Since the equation of the curve remains same, if x is replaced by $-x$ and y by $-y$. The curve is symmetric about opposite quadrants.

Example :- Trace the curve

$$y^2(2a-x) = x^3$$

[2011, 2014]

Solution :- We have $y^2 = \frac{x^3}{2a-x}$ ...(1)

1. Origin :- Equation does not contain any constant term. It passes through origin.

2. Tangent at origin :- $2ay^2 - xy^2 = x^3$

so put $2ay^2 = 0 \Rightarrow y^2 = 0$

$y=0, y=0$ are two tangents at origin.

3. Symmetry about x-axis :- $\because$ the curve contains only even powers of

y, it is symmetric about x-axis.

4. Cusp :- As we have seen, curve has two coincident tangent at origin. Origin is a cusp.

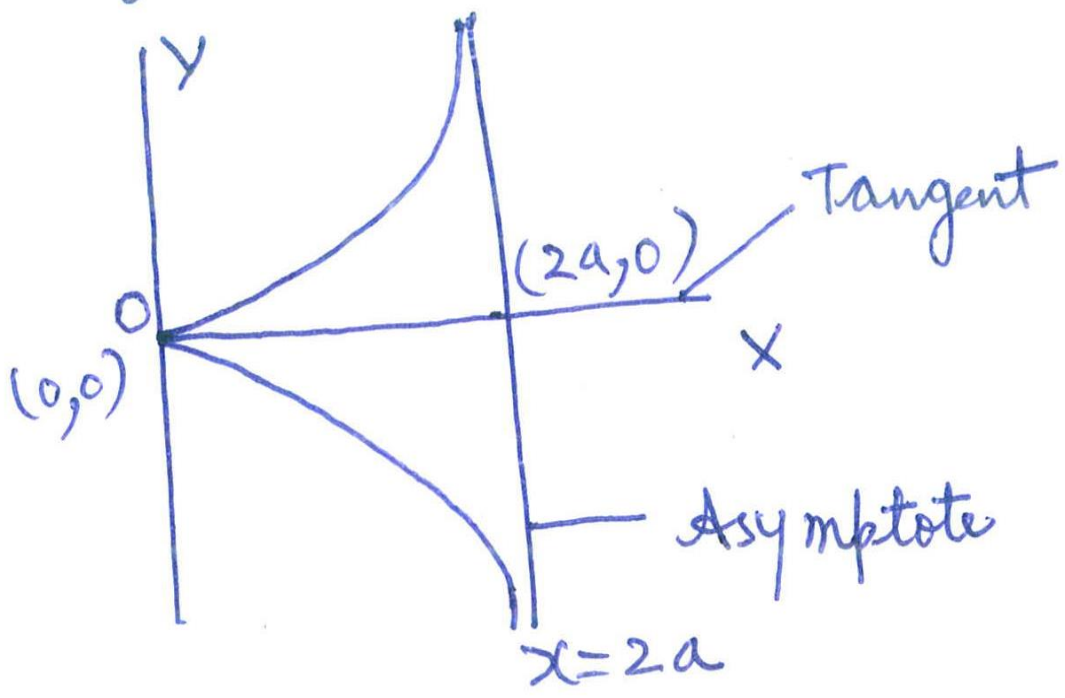
5. Asymptote parallel to y-axis :-

$$(2a-x)y^2 - x^3 = 0$$

$2a - x = 0 \Rightarrow x = 2a$ is the asymptote parallel to y-axis.

6. Region of absence of curve :- y^2 becomes negative on putting $x > 2a$ or $x < 0$, therefore, the curve does not exist for $x < 0$ and $x > 2a$.

So, a sketch of the curve will be



Example :- Trace the curve $y = x(x^2-1)$ [2012]

Solution :- 1. Origin :- Curve passes trough the origin.

2. Tangent at the origin :- $y + x - x^3 = 0$, by putting lowest degree term to zero

$y + x = 0 \Rightarrow y = -x$ is the tangent.

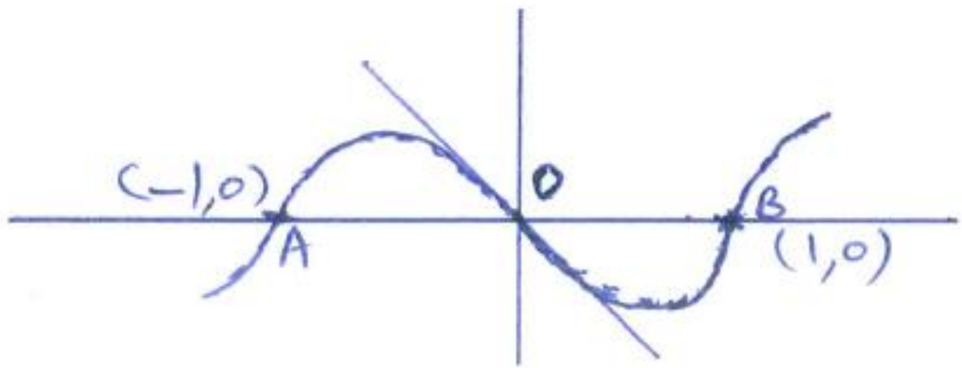
3. Symmetry :- when x is replaced by $-x$ and y by $-y$ then equation of the curve remains same, so symmetry in opposite quadrants.

4. Point of Intersection with x-axis :- Put $x(x^2-1) = 0$

$x = 0, \; x = 1, \; x = -1$

5. Asymptote: There is no asymptote

Sketch of the curve



Example : Trace the curve $ay^2 = x^2(a-x)$

Solution: we have,

$$ay^2 = x^2(a-x) \qquad \text{(i)}$$

1) **Symmetry:** Since the equation (i) contains only even power of y,
   ∴ it is symmetric about the x-axis.

   It is not symmetric about y-axis since it does not contain even power of x.

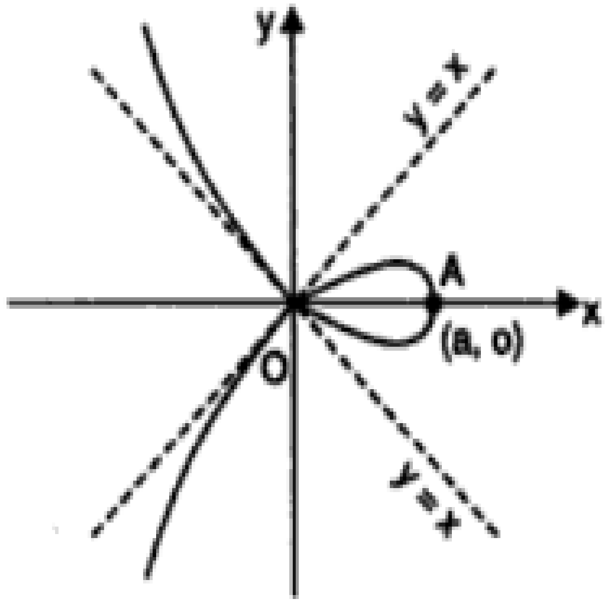
2) **Origin:** Since constant term is absent in (i), it passes through origin.
3) **Intersection with x-axis:**
   Putting y=0 in (i), we get x=a.
   ∴ Curve cuts the x-axis at (a, 0).
4) **Tangent:** The equation of the tangent at origin is obtained by equating to zero the lowest degree term of the equation (i).

$$ay^2 = ax^2.$$

$\Rightarrow y^2 = x^2$

$\Rightarrow y = \pm x$

There are two tangents $y = \pm x$ at the origin to the given curve.



Example : Trace $y^2(a^2+x^2) = x^2(a^2-x^2)$

Solution: Here we have,

$$y^2(a^2+x^2) = x^2(a^2-x^2) \qquad \text{(i)}$$

1) **Origin:** The equation of the given curve does not contain constant term, therefore, the curve passes through origin.
2) **Symmetric about axes:** The equation contains even powers of x as well as y, so the curve is symmetric about both the axes.
3) **Point of intersection with x-axis:** On putting y=0 in the equation, we get

$$x^2(a^2-x^2) = 0, \;\; x = \pm a, 0, 0$$

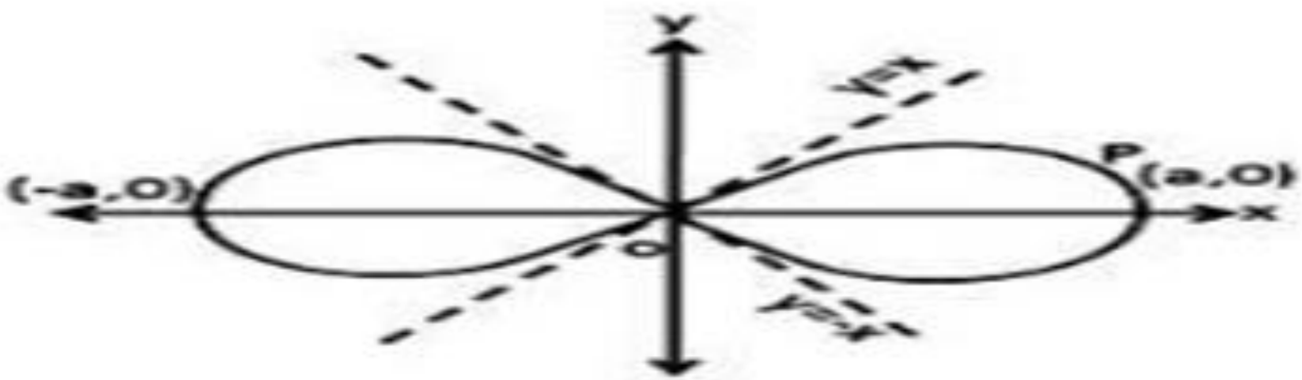
4) **Tangent at the origin:** Equation of the tangent is obtained by equating to zero the lowest degree term.

$$a^2y^2 - a^2x^2 = 0 \Rightarrow \; y = \pm x$$

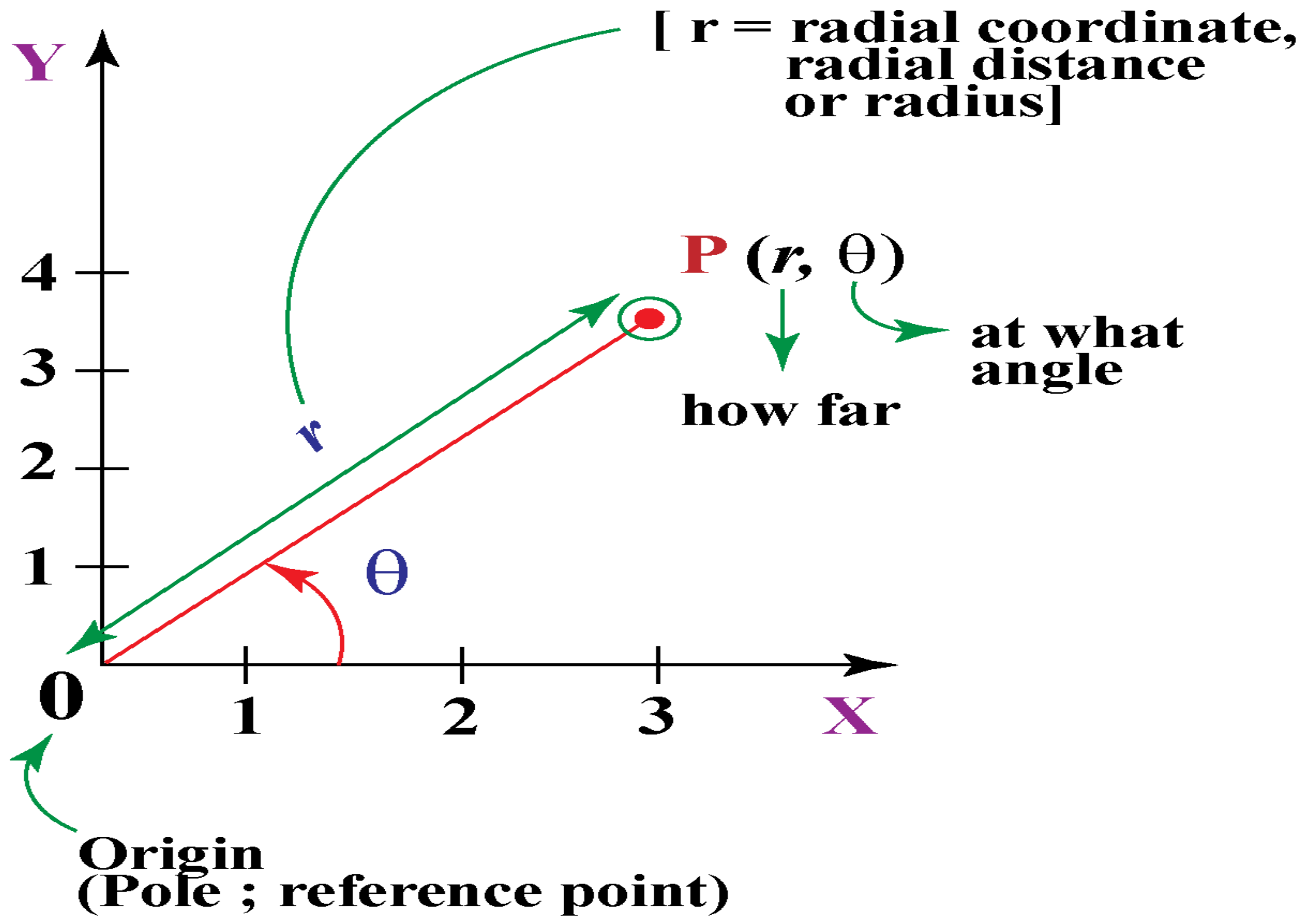
There are two tangents $y = x$ and $y = -x$ at the origin.

5) **Node:** Origin is the node, since, there are two real and different tangents at the origin.

6) **Region of absence of the curve:** For values of $x > a$ and $x < -a$, y^2 becomes negative, hence, the entire curve remains between $x = -a$ and $x = a$.



Polar Coordinate System



TRACING OF POLAR CURVES

The following steps may be taken while tracing the curves in polar co-ordinates :

1. Symmetry :

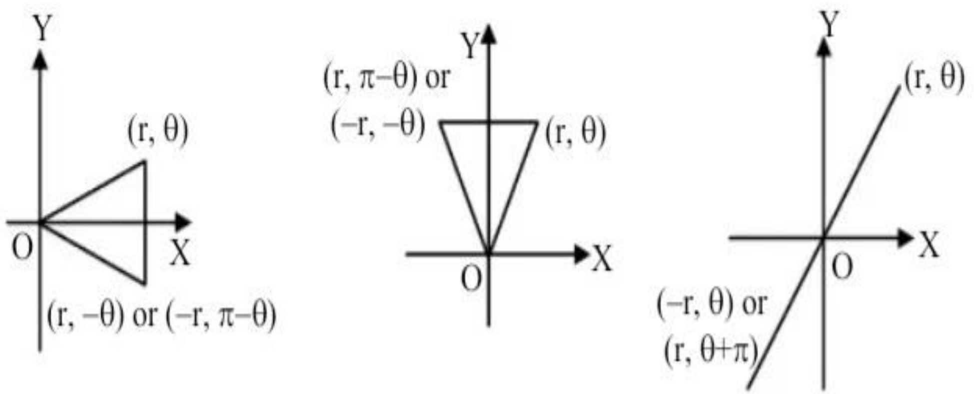
(i) The curve is symmetrical about the x-axis (initial line, $\theta = 0$) if the equation is unchanged when θ is replaced by $-\theta$, or the pair (r, θ) by the pair $(-r, \pi-\theta)$.

e.g. $r \cos\theta = a \sin^2\theta$.

(ii) The curve is symmetrical about the y-axis (the ray $\theta = \pi/2$) if the equation is unchanged when θ is replaced by $\pi-\theta$, or by the pair $(-r, -\theta)$.

e.g. $r\theta = a$.

(iii) The curve is symmetrical about the pole if the equation is unchanged when r is replaced by $-r$, or θ is replaced by $\theta + \pi$.



2. Origin (or the Pole) : Check whether the curve passes through the pole or not. For this put $r = 0$. If we get some real value of θ, then the curve passes through the pole. If we cannot find real value of θ for which $r = 0$, the curve does not pass through the pole. i.e. if on putting $r = 0$, we get $\theta = \theta_1, \theta_2, \ldots$ Where $\theta_1, \theta_2, \ldots$ are all real numbers, then the curve passes through the pole and $\theta = \theta_1$, $\theta = \theta_2 \ldots$ Are the tangents at the pole. But if on putting $r = 0$, we do not get any real value of θ, then the curve does not pass through the pole.

e.g.

For the curve $r^2 = a^2 \cos 2\theta$, for $r = 0$, we get

$$\cos 2\theta = 0 \quad \text{or} \quad \theta = \pm\frac{\pi}{4} \text{ (real values of } \theta)$$

3. Asymptotes : If $r \to \infty$ as $\theta \to \theta_1$ (finite value), then there is an asymptote and we find it as follows : If α is a root of the equation $f(\theta) = 0$, then $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$ is an asymptote of the curve

$\frac{1}{r} = f(\theta)$.

4. Points of Intersection : Find the points of intersection with the lines $\theta = 0$, $\theta = \frac{\pi}{6}, \theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}$ and so on, or we make the table of the values of r corresponding to some suitable values of θ (especially for those values of θ for which the curve is symmetrical).

5. Region : Solve the equation for r and consider how r varies as θ increases from 0 (or some convenient value θ_1) to $+\infty$ and also as θ diminishes from 0 (or θ_1) to $-\infty$.

Find the regions in which the curve does not lie. This can be checked as follows :

(i) If r is imaginary, say for $\alpha<\theta<\beta$, then no portion of the curve lies between the rays $\theta = \alpha$ and $\theta = \beta$.

(ii) If r_{max} is a for all real values of θ, then the whole of the curve lies within a circle of radius a, and if r_{min} is b, the whole of the curve lies outside the circle of radius b.

6. Special Points : Find ϕ(i.e., angle between the tangent and the radius vector) at $P(r, \theta)$ using the relation $\tan\phi = r\frac{d\theta}{dr}$

Find the points where $\phi = 0$ or $\frac{\pi}{2}$.

Also if $\frac{dr}{d\theta}$ is +ve, then r increases as θ increases and if $\frac{dr}{d\theta}$ is –ve, then r decreases as θ decreases.

Example :- Trace the curve $r^2 = a^2 \cos 2\theta$

[2014, 2019]

Solution :- 1. Symmetry :- (i) Since the equation of the curve remains unchanged, if θ is replaced by $-\theta$. The curve is symmetric about initial line.

(ii) $$r^2 = a^2 \cos 2(\pi - \theta) = a^2 \cos 2\theta$$

When the angle θ is replaced by $(\pi - \theta)$, the curve remains unchanged. The curve is symmetric about $\theta = \frac{\pi}{2}$

(iii) Since $$(-r)^2 = a^2 \cos 2(\pi + \theta)$$

$$r^2 = a^2 \cos 2\theta$$

So, the curve is symmetric about opposite quadrants.

2. Origin or pole :- Put $r = 0$ in the equation of the curve, we get

$$\cos 2\theta = 0$$

$$2\theta = \pm \frac{\pi}{2} \Rightarrow \theta = \pm \frac{\pi}{4}.$$

The curve passes through the pole and the tangents at the pole are

$$\theta = \pm \frac{\pi}{4}.$$

Also when $\theta = 0$, $r = \pm a$

the curve meets the initial line at $(\pm a, 0)$

3. Asymptotes:- The curve has no asymptote.

4. Value of ϕ :- $\phi = \frac{\pi}{2} + 2\theta$

when $\theta = 0$, $\phi = \frac{\pi}{2}$

and $r = \pm a$

hence at the points $(a, 0)$ and $(-a, 0)$ the tangents are perpendicular to initial line.

5. Special points and Region

$\because r = a\sqrt{\cos 2\theta}$, $\frac{dr}{d\theta} = \frac{-a\sin 2\theta}{\sqrt{\cos 2\theta}}$

For $0 < \theta < \frac{\pi}{4}$, $\frac{dr}{d\theta}$ is negative

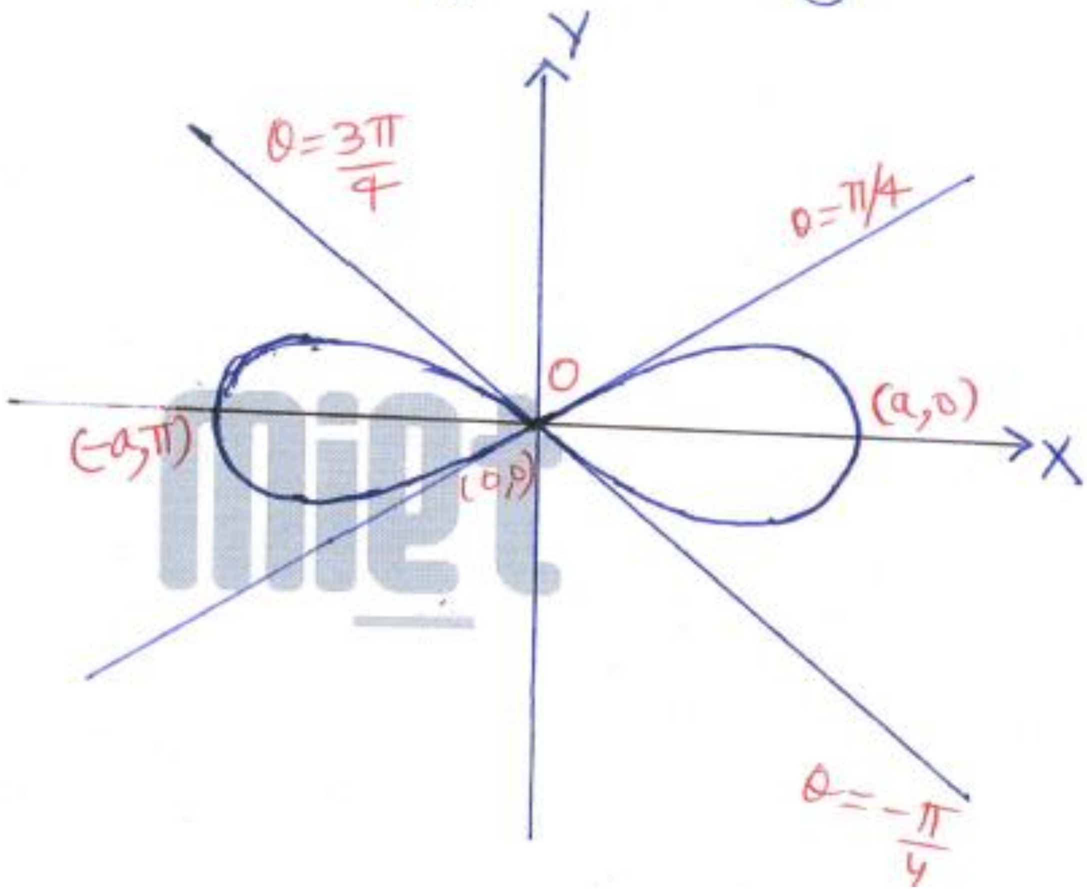
(r decreases in this range)

For $\frac{3\pi}{4} < \theta < \pi$, $\frac{dr}{d\theta}$ is positive

(r increases in this range)

For $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, r imaginary, hence no portion of the curve lies between $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

Thus, we can trace the part of the curve about the initial line. The part of the curve below the initial line can be traced by symmetry.

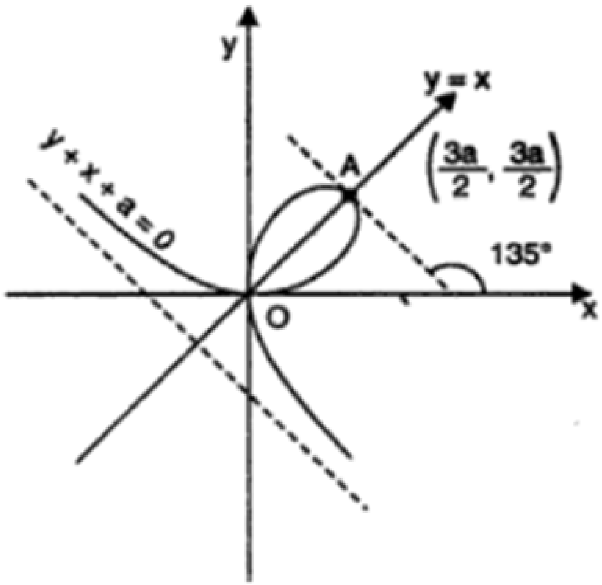


Practice Questions

Q 1.Trace the curve

$x^3 + y^3 = 3axy,\ a > 0$

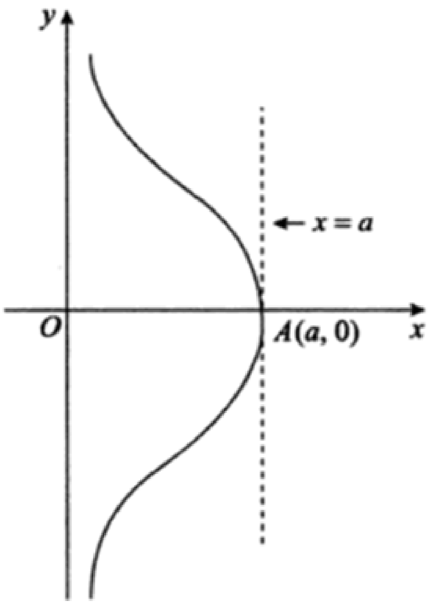
Ans.



Q 2. Trace the curve

$xy^2 = 4a^2 (a - x)$

Ans.

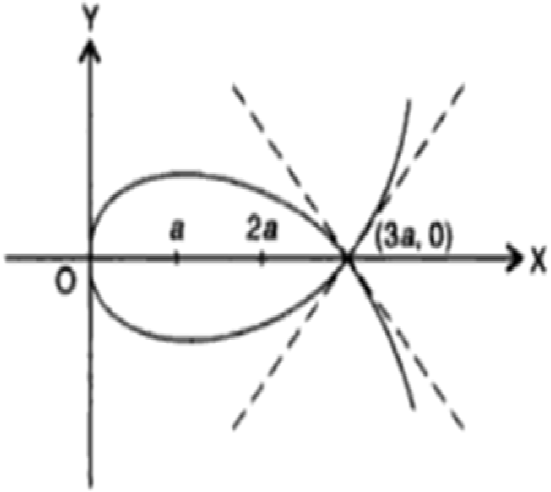


Practice Questions

Q 3. Trace the curve

$9ay^2 = x(x - 3a)^2.$

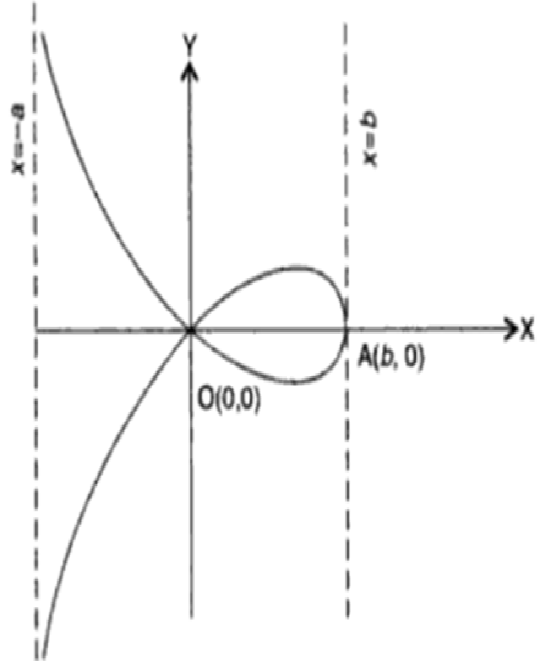
Ans.



Q 4. Trace the curve

$y^2 (a + x) = x^2 (b - x)$

Ans.



Practice Question

Q.5 Trace the curve r=a(1+cos θ)

Ans.