Lecture 21

Taylor's and Maclaurin's Theorem for a function of one variable

Maclaurin's theorem.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

Example *Expand* $\log(1+x)$ *in powers of* x*. Then find series for* $\log_e\left(\frac{1+x}{1-x}\right)$ *and hence determine the value of* $\log_e\left(\frac{11}{9}\right)$ *upto five places of decimal.* (M.T.U. 2012)

Sol. Let $f(x) = \log(1+x)$ $\quad\therefore\quad$ $f(0) = \log 1 = 0$

then $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\quad\therefore\quad$ $f'(0) = 1$

$f''(x) = (-1)(1+x)^{-2}$ $\quad\therefore\quad$ $f''(0) = -1$

$f'''(x) = (-1)(-2)(1+x)^{-3}$ $\quad\therefore\quad$ $f'''(0) = 2$

$f^{iv}(x) = (-1)(-2)(-3)(1+x)^{-4}$ $\quad\therefore\quad$ $f^{iv}(0) = -2 \times 3 = -6$

and so on.

Putting these values in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots$$

$$\therefore \qquad \log(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Changing x into $-x$, we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$$

Now, $\log(1+x) - \log(1-x)$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right]$$

$$\therefore \qquad \log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right]$$

Putting $x = \frac{1}{10}$ in the above result, we get

$$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{5}\left(\frac{1}{10}\right)^5 + \frac{1}{7}\left(\frac{1}{10}\right)^7 + \ldots\right] = 0.20067$$

Example. *Find the values of a and b such that the expansion of* $\log(1+x) - \frac{x(1+ax)}{(1+bx)}$ *in ascending powers of x begins with the term* x^4 *and hence find this term.*

(M.T.U. 2013)

Sol. $\log(1+x) - \frac{x(1+ax)}{(1+bx)} = \log(1+x) - x(1+ax)(1+bx)^{-1}$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - (x + ax^2)(1 - bx + b^2x^2 - b^3x^3 + \dots)$$

| Using binomial expansion

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - [x + (a-b)x^2 - b(a-b)x^3 + b^2(a-b)x^4 - \dots]$$

$$= \left(-\frac{1}{2} - a + b\right)x^2 + \left\{\frac{1}{3} + b(a-b)\right\}x^3 - \left\{\frac{1}{4} + b^2(a-b)\right\}x^4 + \dots \quad \dots(1)$$

According to the question,

$$-\frac{1}{2} - a + b = 0 \quad \Rightarrow \quad a - b = -\frac{1}{2} \qquad \text{...(2)}$$

and $\frac{1}{3} + b(a - b) = 0 \quad \Rightarrow \quad \frac{1}{3} - \frac{b}{2} = 0 \quad \Rightarrow \quad b = \frac{2}{3}$

$\therefore$ From (2), $a = \frac{1}{6}$

$$\text{Required term} = -\left[\frac{1}{4} + b^2(a - b)\right] x^4$$

$$= -\left[\frac{1}{4} + \frac{4}{9}\left(-\frac{1}{2}\right)\right] x^4 = -\frac{1}{36}\, x^4.$$

EXPANSION OF FUNCTIONS OF ONE VARIABLE

Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^n(x) + \ldots$$

TAYLOR'S THEOREM

Working Rule

Step 1. Put the given function equal to $f(x+h)$.

Step 2. Put $h = 0$ and write $f(x)$.

Step 3. Differentiate $f(x)$ a number of times and obtain $f'(x), f''(x), f'''(x)$, etc. ...

Step 4. Now substitute the values of $f(x), f'(x), f''(x)$ ... in

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

Example. (*a*) *Expand* $\sin(x+y)$ *in powers of y and deduce that*

$$\sin(x+y) = \sin x \cos y + \sin y \cos x.$$

(*b*) *Obtain the value of* $\sin 31°$ *correct to four places of decimals.*

Sol. (*a*) (*i*) Put $f(x+y) = \sin(x+y)$

(*ii*) Making $y = 0$, we have $f(x) = \sin x$

(*iii*) $\therefore \quad f'(x) = \cos x; \qquad f''(x) = -\sin x,$

$f'''(x) = -\cos x; \qquad f^{iv}(x) = \sin x, \ldots,$ and $f^{n}(x) = \sin(x + n\pi/2)$

Now, $f(x+y) = f(x) + yf'(x) + \dfrac{y^2}{2!}f''(x) + \ldots$

$$\therefore \sin(x+y) = \sin x + y\cos x - \frac{y^2}{2!}\sin x - \frac{y^3}{3!}\cos x + \frac{y^4}{4!}\sin x + \ldots \qquad \ldots(1)$$

$$= \sin x\left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right] + \cos x\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right]$$

$$= \sin x \cos y + \cos x \sin y.$$

(b) Put $x = \frac{\pi}{6}$ and $y = 1° = \frac{\pi}{180} = .0175$ radians in (1), Then

$$\sin 31° = \sin\frac{\pi}{6} + \frac{\pi}{180}\cos\frac{\pi}{6} - \left(\frac{\pi}{180}\right)^2 \frac{1}{2!}\sin\frac{\pi}{6} + \ldots = 0.5151$$

correct to four places of decimal.

ANOTHER FORM OF TAYLOR'S SERIES

$f(x)$ is expressed as a series in ascending integral powers of $x - a$.

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots$$

Example. *Expand sin x in ascending powers of* $\left(x-\frac{\pi}{2}\right)$.

Sol. Here $f(x) = \sin x$

$$f(x) = f\left(\frac{\pi}{2}+x-\frac{\pi}{2}\right) = f(a+h), \text{ where } a = \frac{\pi}{2} \text{ and } h = x-\frac{\pi}{2}$$

$$f(x) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f^{iv}(a) + \ldots$$

Putting these values of a and h, we get

$$\sin x = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right) + \ldots$$

But

$$f(x) = \sin x \qquad \therefore \qquad f\left(\frac{\pi}{2}\right) = 1$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{2}\right) = 0$$

$$f^{iv}(x) = \sin x \qquad f^{iv}\left(\frac{\pi}{2}\right) = 1 \text{ and so on.}$$

Putting these values in (1), we get

$$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \ldots$$

Example. *Expand $\log x$ in powers of $(x-1)$ by Taylor's theorem and hence find the value of $\log_e (1.1)$.*

Sol. Here $f(x) = \log x, \quad a = 1 \quad \therefore \quad f(1) = 0$

Now,

$$f'(x) = \frac{1}{x}, \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2$$

$$f^{(iv)}(x) = \frac{-6}{x^4}, \qquad f^{(iv)}(1) = -6 \text{ and so on}$$

We know that,

$$f(x) = f(1) + (x-1)\,f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \ldots$$

$$\Rightarrow \quad \log x = 0 + (x-1)\,.\,1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}\,.\,2 + \frac{(x-1)^4}{4!}(-6) + \ldots$$

$$\Rightarrow \quad \log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$$

Putting $x = 1.1$, we get

$$\log (1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \ldots$$

$$= 0.095305$$

to expand functions like $(\sin^{-1} x)^2$, $e^{a \sin^{-1} x}$, $\sin (m \sin^{-1} x)$, $\cos (m \sin^{-1} x)$ etc. by the method of differential equations.

Working rule:

1. Put the given function equal to y.
2. Find $y_1 = \dfrac{dy}{dx}$.

Then (*i*) Take LCM (if possible).

(*ii*) Square both sides if square roots are there.

(*iii*) Try to get y in RHS (if possible).

3. Again differentiate both sides w.r.t. x to get an equation in y_2, y_1 and y
4. Differentiate both sides n times w.r.t. x by Leibnitz Theorem.
5. Put $x = 0$ in equations of steps 1, 2, 3, 4.
6. Put $n = 1, 2, 3, 4$ in last equation of step 5.
7. Discuss the two cases when n is even and when n is odd.

Example. *If* $y = (\sin^{-1} x)^2$, *show that*

(a) $(1 - x^2)\, y_2 - xy_1 = 2$

(b) $$\frac{(\sin^{-1} x)^2}{2} = \frac{x^2}{2\,!} + \frac{2^2}{4\,!} \cdot x^4 + \frac{2^2 . 4^2}{6\,!} x^6 + \ldots$$

Sol. Let $$y = (\sin^{-1} x)^2 \quad \text{...(1)}$$

$$\therefore \quad y_1 = 2 \sin^{-1} x \times \frac{1}{\sqrt{1-x^2}} \quad \text{...(2)}$$

$$\sqrt{1-x^2} \, . \, y_1 = 2\sqrt{y}$$

$$(1 - x^2) \, {y_1}^2 = 4y.$$

Differentiating again,

$$(1 - x^2) \, . \, 2y_1 \, y_2 + {y_1}^2 \, (- 2x) = 4y_1$$

Dividing by $2y_1$, $$(1 - x^2)y_2 - xy_1 = 2 \quad \text{...(3)}$$

which proves part (a).

Differentiating (3), n times by Leibnitz's Theorem.

$$(1-x^2)\,y_{n+2} + {}^nC_1\,y_{n+1}\,(-2x) + {}^nC_2\,y_n\,(-2) - xy_{n+1} - {}^nC_1\,y_n\,.\,1 = 0$$

$$(1-x)^2\,y_{n+2} - 2nxy_{n+1} - n(n-1)\,y_n - xy_{n+1} - ny_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n-1)\,xy_{n+1} = n^2y_n \qquad \ldots(4)$$

Putting $x = 0$, in (1), (2), (3), (4), we get

$$y = 0, \quad y_1 = 0, \quad y_2 = 2$$

$$(y_{n+2})_0 = n^2(y_n)_0$$

Putting $n = 1, 2, 3, 4$ in the last equation, we get

$$y_3 = 1^2\,y_1 = 0$$

$$y_4 = 2^2\,y_2 = 2^2\,.\,2 = 2.2^2$$

$$y_5 = 3^2\,y_3 = 0$$

$$y_6 = 4^2\,y_4 = 2.2^2\,.\,4^2$$

....

$\therefore$ By Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$(\sin^{-1} x)^2 = 0 + x(0) + \frac{x^2}{2!} \,.\, 2 + \frac{x^3}{3!} \,.\, (0) + \frac{x^4}{4!}\,(2.2^2) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(2.2^2 \,.\, 4^2) + \ldots$$

$$\frac{(\sin^{-1} x)^2}{2} = \frac{x^2}{2!} + \frac{2^2}{4!}x^4 + \frac{2^2 \,.\, 4^2}{6!}x^6 + \ldots$$ which proves part (*b*).

HOME WORK

1. *Expand a^x and e^x in powers of x by Maclaurin's theorem.*

2. Expand $4x^2 + 7x + 5$ in powers of $x - 3$.

3. *Expand $\sin^{-1} x$ in ascending powers of x.*

4. Expand $e^{2x} \sin x$ in ascending powers of x up to x^5.

Lecture 21

Taylor's and Maclaurin's Theorem for a function of two variables

Taylor's Theorem for a function of Two Variables

This is used to expand $f(x,y)$ in the neighbourhood of (a,b) [or extend $f(x,y)$ in powers of $(x-a)$ and $(y-b)$]

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{\underline{|2}}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

Maclaurin's Theorem for a function of Two Variables

This is used to expand $f(x,y)$ in the neighbourhood of origin $(0,0)$ [or expand $f(x,y)$ in powers of x and y]

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{\lfloor 2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \ldots$$

Note- Maclaurin's theorem is obtained by Taylor's theorem by putting $a=0$ and $b=0$.

MIET GROUP OF INSTITUTIONS

Q.1. State the Taylor's Theorem for two variables.

Sol. Please see the statement above. (2018-19)

Q2. Expand $e^x \log(1+y)$ in the powers of x and y upto terms of third degree. (2014-15)

Sol. $f(x,y) = e^x \log(1+y)$, $f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$, $f_x(0,0) = 0$

$f_y(x,y) = \dfrac{e^x}{1+y}$, $f_y(0,0) = 1$

(Note $e^0 = 1$, $\log 1 = 0$)

$$f_{xx}(x,y) = e^x \log(1+y) \quad , \quad f_{xx}(0,0) = 0$$

$$f_{xy}(x,y) = \frac{e^x}{1+y} \quad , \quad f_{xy}(0,0) = 1$$

$$f_{yy}(x,y) = -\frac{e^x}{(1+y)^2} \quad , \quad f_{yy}(0,0) = -1$$

$$f_{xxx}(x,y) = e^x \log(1+y) \quad , \quad f_{xxx}(0,0) = 0$$

$$f_{xxy}(x,y) = \frac{e^x}{1+y} \quad , \quad f_{xxy}(0,0) = 1$$

$$f_{xyy}(x,y) = -\frac{e^x}{(1+y)^2} \quad , \quad f_{xyy}(0,0) = -1$$

$$f_{yyy}(x,y) = \frac{2e^x}{(1+y)^3}, \quad f_{yyy}(0,0) = 2$$

Maclaurin's Theorem is given by

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

$$\therefore e^x \log(1+y) = 0 + [x.0 + y.1] + \frac{1}{2}[x^2.0 + 2xy.1 + y^2(-1)] + \frac{1}{6}[x^3.0 + 3x^2y.1 + 3xy^2(-1) + y^3.2] + \cdots$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \cdots$$

Q3 Express the function $f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$ as Taylor's Series expansion about the point $(1,2)$.

(2016-17, 2017-18)

Sol. Here

$f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$, $f(1,2) = 12$

$f_x(x,y) = 2x - 9$, $f_x(1,2) = -7$

$f_y(x,y) = 6y - 9$, $f_y(1,2) = 3$

$f_{xx}(x,y) = 2$, $f_{xx}(1,2) = 2$

$f_{xy}(1,2) = 0$

$f_{xy}(x,y) = 0$

$f_{yy}(x,y) = 6$, $f_{yy}(1,2) = 6$

Taylor's series about (1, 2) is given by

$$f(x,y) = f(1,2) + [(x-1) f_x(1,2) + (y-2) f_y(1,2)] + \frac{1}{\lfloor 2} [(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2) f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)] + \cdots$$

$$f(x,y) = 12 - 7(x-1) + 3(y-2) + \frac{1}{2}[2(x-1)^2 + 0 + 6(y-2)^2]$$

$$= 12 - 7(x-1) + 3(y-2) + (x-1)^2 + 3(y-2)^2$$

Q.4 Expand $f(x,y) = y^x$ about $(1,1)$ upto second degree terms and hence evaluate $(1.02)^{1.03}$ (2012-13)

Sol.

$f(x,y) = y^x$, $f_x(x,y) = y^x \log y$, $f_y(x,y) = x y^{x-1}$,

$f_{xx}(x,y) = y^x (\log y)^2$, $f_{yy}(x,y) = x(x-1) y^{x-2}$,

$f_{xy}(x,y) = x y^{x-1} \log y + y^x \cdot \frac{1}{y} = x y^{x-1} \log y + y^{x-1}$

at $(1,1)$;

$f(1,1) = 1$, $f_x(1,1) = 0$, $f_y(1,1) = 1$, $f_{xx}(1,1) = 0$, $f_{yy}(1,1) = 0$, $f_{xy}(1,1) = 1$

Taylor's series about $(1,1)$ upto second degree term is

$$f(x,y) = f(1,1) + [(x-1) f_x(1,1) + (y-1) f_y(1,1)] + \frac{1}{2}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)] + \cdots$$

$$y^x = 1 + (y-1) + (x-1)(y-1) + \cdots \quad (1)$$

Now put $y = 1.02$ and $x = 1.03$ in eq (1),

$$(1.02)^{1.03} = 1 + (0.02) + (0.03)(0.02) + \cdots$$

$$= 1 + 0.02 + 0.0006 + \cdots$$

$$\simeq 1.0206 \quad \text{(approximately)}$$

HOME WORK

1. Expand e^{xy} at (1, 1) upto three terms.

Ans. $e\left[1+(x-1)+(y-1)+\frac{1}{2!}\left[(x-1)^2+4(x-1)(y-1)+(y-1)^2\right]\right]$

2. Expand y^x at (1, 1) upto second term. **Ans.** $1+(y-1)+(x-1)(y-1)+\ldots\ldots$

3. Expand $e^{ax}\sin by$ in powers of x and y as far as the terms of third degree. *(U.P. I sem. Jan 2011)*

Ans. $by+abxy+\frac{1}{3!}\left(3a^2bx^2y-b^3y^3\right)+\ldots$

4. Expand $(x^2y+\sin y+e^x)$ in powers of $(x-1)$ and $(x-\pi)$.

Ans. $\pi+e+(x-1)(2\pi+e)+\frac{1}{2}(x-1)^2(2\pi+e)+2(x-1)(y-\pi)$.

Lecture 22

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLES

Maxima and Minima of function of two Variables $z=f(x,y)$

Working Rule:-

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) Solve $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$ simultaneously

Let (a,b); (c,d), ... — be the solution of these equations

Note- These points are called stationary points.

(iii) Find $r=\frac{\partial^2 z}{\partial x^2}=f_{xx}$; $s=\frac{\partial^2 z}{\partial x \partial y}=f_{xy}$; $t=\frac{\partial^2 z}{\partial y^2}=f_{yy}$

(iv) Find $rt-s^2$ for each stationary point, then

(a) If $rt - s^2 > 0$ and $r < 0$ for (a, b), then z has a maximum value at (a, b)

(b) If $rt - s^2 > 0$ and $r > 0$ for (a, b), then z has a minimum value at (a, b)

(c) If $rt - s^2 < 0$ for (a, b) then z has no extreme value at (a, b) and (a, b) is called a saddle point.

Note– (1) A maximum or a minimum value of a function is called its extreme value.

(2) If $rt - s^2 = 0$ then further investigation is required.

Q1. Find the stationary point of (2013-14)

$$f(x,y) = 5x^2 + 10y^2 + 12xy - 4x - 6y + 1.$$

Sol. $\frac{\partial f}{\partial x} = f_x = 10x + 12y - 4$

$\frac{\partial f}{\partial y} = f_y = 20y + 12x - 6$

For stationary point solve $f_x = 0$ and $f_y = 0$

$\Rightarrow \quad 10x + 12y - 4 = 0 \quad$ or $\quad 5x + 6y = 2 \quad$ — ①

and $\quad 20y + 12x - 6 = 0 \quad$ or $\quad 6x + 10y = 3 \quad$ — ②

Multiplying eq ① by 6, eq ② by 5 then subtracting, we get

$$\begin{array}{r} 30x + 36y = 12 \\ 30x + 50y = 15 \\ - \quad\; - \quad\;\; - \\ \hline -14y = -3 \end{array} \Rightarrow y = \frac{3}{14}$$

Put value of y in eq ①,

$$5x + \frac{9}{7} = 2 \Rightarrow 5x = \frac{5}{7} \Rightarrow x = \frac{1}{7}$$

Stationary point is $\left(\frac{1}{7}, \frac{3}{14}\right)$.

Q2. Find the maximum value of the function $f(x, y, z) = z - 2x^2 - 2y^2$ where $3xy - z + 7 = 0$ (2016-17)

Sol. Given $3xy - z + 7 = 0 \Rightarrow z = 3xy + 7$

on putting this value of z, we get

$$f(x, y) = 3xy + 7 - 2x^2 - 2y^2 \quad \text{--- (1)}$$

$$f_x = 3y - 4x \quad , \quad f_y = 3x - 4y$$

For stationary point, solve $f_x = 0$ and $f_y = 0$

$\Rightarrow 3y - 4x = 0$ --- (2) and $3x - 4y = 0$ --- (3)

on solving eq. ② and ③ we get $x=0,\ y=0$

$\therefore$ (0, 0) is the stationary point.

Now

$r = f_{xx} = -4$, $s = f_{xy} = 3$, $t = f_{yy} = -4$

$rt - s^2 = (-4)(-4) - (3)^2 = 16 - 9 = 7 > 0$

$\because rt - s^2 > 0$ and $r < 0$

$\therefore f(x, y)$ has maximum value at (0, 0)

Put $x=0,\ y=0$ in eq ①, Maximum value is

$\text{Max. } f = 0 + 7 - 0 - 0 = 7$

Ans

Q4. Find the extreme values of function $x^3 + y^3 - 3axy$.

Sol. Here $f(x, y) = x^3 + y^3 - 3axy$ (2018-19)

$f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = f_{yy} = 6y$

Now $f_x = 0$ and $f_y = 0$

$\Rightarrow$ $x^2 - ay = 0$ — ① and $y^2 - ax = 0$ — ②

From ①, $y = \frac{x^2}{a}$ — ③, Put this value of y in ②,

$\frac{x^4}{a^2} - ax = 0$ or $x(x^3 - a^3) = 0$ or $x = 0, a$

When $x = 0$, $y = 0$; when $x = a$, $y = a$ (using eq. ③)

∴ There are two stationary points $(0,0)$ and (a, a).

Now $rt - s^2 = 36xy - 9a^2$

at $(0,0)$ $rt - s^2 = -9a^2 < 0$ There is no extreme value at $(0,0)$

at (a,a) $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$\Rightarrow$ $f(x,y)$ has extreme value at (a,a)

Now at (a,a), $r = 6a$

(i) If $a>0$, $r>0$ so that $f(x,y)$ has a minimum value at (a,a)

Minimum value $= a^3 + a^3 - 3a^3 = -a^3$

(ii) If $a<0$, $r<0$ so that $f(x,y)$ has a maximum value at (a,a)

Maximum value $= (-a)^3 + (-a)^3 - 3(-a)(-a)(-a)$

$= -a^3 - a^3 + 3a^3 = a^3$. Ans

Q5. Find the stationary point of $f(x,y) = x^3 + y^3 + 3axy$, $(a>0)$

Sol. Proceed as above (2018-19)

Ans (0,0) and $(-a,-a)$ are stationary points.

Example

Examine the following surface for high and low points $z = x^2 + xy + 3x + 2y + 5$.

Sol.

$$\frac{\partial z}{\partial x} = 2x + y + 3; \quad \frac{\partial z}{\partial y} = x + 2$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2; \; s = \frac{\partial^2 z}{\partial x \partial y} = 1; \; t = \frac{\partial^2 z}{\partial y^2} = 0$$

For the maximum or minimum we must have

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0.$$

From $\frac{\partial z}{\partial x} = 0$, we get $2x + y + 3 = 0$ ...(*i*)

From $\frac{\partial z}{\partial y} = 0$, we get $x + 2 = 0$. ...(*ii*)

From (*ii*), we get $x = -2$ and $\therefore$ from (*i*) $y = -3 + 4 = 1$.

Hence, the solution is $x = -2$, $y = 1$ and for these values we have $r = 2$, $s = 1$, $t = 0$.

$\therefore \; rt - s^2 = (2)(0) - (1)^2 = -1 < 0.$

$\therefore$ There is neither maximum nor minimum at $x = -2$, $y = 1$.

Example

Test the function $f(x, y) = x^3y^2(6 - x - y)$ for maxima and minima for points not at the origin.

Sol. Here $f(x, y) = x^3y^2(6 - x - y) = 6x^3y^2 - x^4y^2 - x^3y^3$

$\therefore$

$$f_x = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 12x^3y - 2x^4y - 3x^3y^2$$

$$r = f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(6 - 2x - y)$$
$$s = f_{xy} = 36x^2y - 8x^3y - 9x^2y^2 = x^2y(36 - 8x - 9y)$$
$$t = f_{yy} = 12x^3 - 2x^4 - 6x^3y = x^3(12 - 2x - 6y)$$

Now, $f_x = 0$

$$\Rightarrow \quad x^2y^2(18 - 4x - 3y) = 0 \qquad \text{...(1)}$$

and $f_y = 0$

$$\Rightarrow \quad x^3y(12 - 2x - 3y) = 0 \qquad \text{...(2)}$$

From (1) and (2),

$$4x + 3y = 18$$

$$2x + 3y = 12 \quad \text{and} \quad x = 0 = y$$

Solving, we get $x = 3, y = 2$ and $x = 0 = y$. Leaving $x = 0 = y$, we get $x = 3, y = 2$.

Hence, (3, 2) is the only stationary point under consideration.

Now, $rt - s^2 = 6x^4y^2(6 - 2x - y)(12 - 2x - 6y) - x^4y^2(36 - 8x - 9y)^2$

At (3, 2) $rt - s^2 = (+)\text{ ve } (> 0)$

Also, $r = 6(3)(4)(6 - 6 - 4) = (-)\text{ ve } (< 0)$

$\therefore$ $f(x, y)$ has a maximum value at (3, 2).

Example

Examine for minimum and maximum values: sin x + sin y + sin (x + y).

Sol. Here

$$f(x, y) = \sin x + \sin y + \sin (x + y)$$

$$f_x = \cos x + \cos (x + y)$$

$$f_y = \cos y + \cos (x + y)$$

$$r = f_{xx} = -\sin x - \sin (x + y)$$

$$s = f_{xy} = -\sin (x + y)$$

$$t = f_{yy} = -\sin y - \sin (x + y)$$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow$ $\cos x + \cos (x + y) = 0$ ...(1) and $\cos y + \cos (x + y) = 0$ ...(2)

Subtracting equation (2) from (1),

$$\cos x - \cos y = 0 \text{ or } \cos x = \cos y \quad \therefore \quad x = y$$

From (1), $\cos x + \cos 2x = 0$

or $\cos 2x = -\cos x = \cos (\pi - x)$

or $2x = \pi - x \quad \therefore \quad x = \dfrac{\pi}{3}$

$\therefore \quad x = y = \frac{\pi}{3}$ is a stationary point.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $\quad r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}, \; s = \frac{\sqrt{3}}{2}, \; t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore \quad rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

Also $r < 0$

$\therefore \quad f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$.

HOME WorK

Find the stationary points of the following functions

1. $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ **Ans.** $\left(\frac{2}{3}, -\frac{4}{3}\right)$; Minimum

2. $f(x, y) = x^3 y^2 (1 - x - y)$ **Ans.** $\left(\frac{1}{2}, \frac{1}{3}\right)$; Maximum

3. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. **Ans.** (6, 0), (4, 0)

4. $f(x, y) = x^2 + 2xy + 2y^2 + 2x + 3y$ such that $x^2 - y = 1$. **Ans.** $\left(-\frac{3}{4}, -\frac{7}{16}\right), -\frac{155}{128}$.

Lecture 23

Lagrange's method of undetermined multipliers

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

To find the maximum or minimum values of a function of three (or more) variables, when the variables are not independent but are connected by some given relation, we try to convert the given function to the one, having least number of independent variables with the help of the given relation.

When this procedure is not practicable, we use Lagrange's method.

Let $f(x, y, z)$ be a function of x, y, z which is to be examined for maximum or minimum value.

Let the variables x, y, z be connected by the relation $\phi(x, y, z) = 0$ ...(1)

For $f(x, y, z)$ to have a maximum or minimum value, the necessary condition is

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

$$\therefore \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0 \quad \text{...(2)}$$

Also, from (1), taking differentials, we get $\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0$ ...(3)

Multiplying (3) by a parameter λ and adding to (2), we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}\right)dz=0$$

This equation will hold good if $\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}=0$, $\frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}=0$, $\frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}=0$.

These equations together with equation (1), give the values of x, y, z and λ for a maximum or minimum.

Lagrange's method does not enable us to find whether there is a maximum or minimum. This fact is determined from the physical considerations of the problem.

Note.

The above equations can be easily obtained by considering Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

and considering the stationary values of $F(x, y, z)$. For stationary values of $F(x, y, z)$, $dF = 0$.

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0.$$

Q1 Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

(2013-14)

Sol. Let x, y, z be three parts of 24, such that

$$x + y + z = 24 \quad \text{--- (1)}$$

We have to maximize $f(x, y, z) = z \cdot y^2 x^3$

(where z is Ist, y is IInd and x is IIIrd part of 24)

Now Lagrange's Function → Relation

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

$F(x,y,z) = zy^2x^3 + \lambda(x+y+z-24)$

For stationary point, $dF = 0$

$\Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$

$\Rightarrow [3zy^2x^2 + \lambda ..]dx + [2zyx^3 + \lambda]dy + [y^2x^3 + \lambda ..]dz = 0$

$\Rightarrow 3zy^2x^2 + \lambda = 0$ —— (2)

$2zyx^3 + \lambda = 0$ —— (3)

$y^2x^3 + \lambda .. = 0$ —— (4)

Now multiplying eq.(2) by x, (3) by y, (4) by z and adding

$$6zy^2x^3 + \lambda(x+y+z) = 0 \qquad (\text{using } (1),\ x+y+z=24)$$

$$\Rightarrow \boxed{\lambda = -\frac{zy^2x^3}{4}}$$

eq (2) $\Rightarrow 3zy^2x^2 - \frac{zy^2x^3}{4} = 0 \Rightarrow zy^2x^2\left(3-\frac{x}{4}\right) = 0 \Rightarrow 3 = \frac{x}{4}$

$\Rightarrow \boxed{x = 12}$

eq (3) $\Rightarrow 2zyx^3 - \frac{zy^2x^3}{4} = 0 \Rightarrow zyx^3\left(2-\frac{y}{4}\right) = 0 \Rightarrow 2 = \frac{y}{4} \Rightarrow \boxed{y = 8}$

eq (4) $\Rightarrow y^2x^3 - \frac{zy^2x^3}{4} = 0 \Rightarrow y^2x^3\left(1-\frac{z}{4}\right) = 0 \Rightarrow 1 = \frac{z}{4} \Rightarrow \boxed{z = 4}$

Hence required parts of 24 are 4, 8, 12.

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Q2. Divide a number into three parts such that the product of first, square of the second and the cube of third is maximum. (2016-17)

Sol. Hint: Let number is a, then proceed as above.

Answer is $\frac{a}{6}, \frac{a}{3}, \frac{a}{2}$

QUESTION – 3

Use the method of Lagrange's multiplier to find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol. Let

$$l = 2x$$
$$b = 2y$$
$$h = 2z$$

$\therefore$ $$V = 8xyz \quad \text{...(i)}$$

and $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \text{...(ii)}$$

For largest volume, from (*i*) and (*ii*), we get

$$dV = yz \cdot dx + xz \cdot dy + xy \cdot dz = 0 \quad \text{...(iii)}$$

and $$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz = 0 \quad \text{...(iv)}$$

Now, equation (*iii*) + λ × equation (*iv*), we get

$$\left(yz + \frac{\lambda}{a^2}x\right) dx + \left(xz + \frac{\lambda}{b^2}y\right) dy + \left(xy + \frac{\lambda}{c^2}z\right) dz = 0$$

$\Rightarrow$ $$yz + \frac{\lambda}{a^2}x = 0, \quad xz + \frac{\lambda}{b^2}y = 0, \quad xy + \frac{\lambda}{c^2}z = 0 \quad \text{...(v)}$$

Multiplying (v) with x, y, z respectively and adding then, we get

$$3xyz + \lambda\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right] = 0$$

$$\Rightarrow \quad 3xyz + \lambda = 0 \quad (\text{using } ii)$$

$$\Rightarrow \quad \lambda = -3xyz$$

Putting the value of λ in any one of equation (v), we get

$$yz - 3xyz \cdot \frac{x}{a^2} = 0 \Rightarrow yz\left(1-\frac{3x^2}{a^2}\right) = 0$$

$$\Rightarrow \quad 1-\frac{3x^2}{a^2} = 0 \Rightarrow x = \frac{a}{\sqrt{3}},$$

Similarly, $$y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

Hence, the largest volume $V = 8 \cdot \dfrac{abc}{3\sqrt{3}}$.

Example

Find the extreme value of $x^2 + y^2 + z^2$, subjected to the condition $xy + yz + zx = p$.

Solution: Let $f = x^2 + y^2 + z^2$ and $\phi = xy + yx + zx - p$.

Then for maximum or minimum, we have

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(y + z) = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(x + z) = 0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda(x + y) = 0 \qquad \text{(iii)}$$

Multiplying equation (i) by x, equation (ii) by y and equation (iii) by z and adding we get.

$2(x^2 + y^2 + z^2) + 2\lambda\,(xy + yz + zx) = 0$

$\Rightarrow 2f + 2\lambda p = 0$

or $\lambda = -\dfrac{f}{p}$

on putting this value of λ in (i) (ii) and (iii), we get

$$2x - \frac{f}{p}(y + z) = 0$$

$$2y - \frac{f}{p}(x + z) = 0$$

$$2z - \frac{f}{p}(x + y) = 0$$

or $2x - 2y + \frac{f}{p}(x - y) = 0$

$$\left(\frac{f}{p} + 2\right)(x - y) = 0$$

$\frac{f}{p} = -2$ and $x = y$

Similarly, we get $y = z$, therefore, $f = 3x^2$

But

$xy + yz + zx = p$

$\Rightarrow 3x^2 = p \Rightarrow x^2 = p/3$

Therefore, extrema occur if $x^2 + y^2 + z^2 = p$ Answer.

Example *Find the maximum value of $u = x^p y^q z^r$ when the variables x, y, z are subject to the condition $ax + by + cz = p + q + r$.*

Solution. Here, we have $u = x^p y^q z^r$ ...(1)

If $\log u = p \log x + q \log y + r \log z$ ...(2)

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x} \Rightarrow \frac{\partial u}{\partial x} = \frac{pu}{x}$$

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{q}{y} \Rightarrow \frac{\partial u}{\partial y} = \frac{qu}{y}$$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{r}{z} \Rightarrow \frac{\partial u}{\partial z} = \frac{ru}{z}$$

$$ax + by + cz = p + q + r$$

$$\phi(x, y, z) = ax + by + cz - p - q - r$$

$$\frac{\partial \phi}{\partial x} = a, \quad \frac{\partial \phi}{\partial y} = b, \quad \frac{\partial \phi}{\partial z} = c$$

Lagranges equations are

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow \frac{pu}{x}+\lambda a = 0 \Rightarrow x = -\frac{pu}{\lambda a}$$

$$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow \frac{qu}{y}+\lambda b = 0 \Rightarrow y = -\frac{qu}{\lambda b}$$

$$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow \frac{ru}{z}+\lambda c = 0 \Rightarrow z = -\frac{ru}{\lambda c}$$

Putting in (2), we have

$$-\frac{pu}{\lambda}-\frac{qu}{\lambda}-\frac{ru}{\lambda} = p+q+r$$

$$-\frac{u}{\lambda}(p+q+r) = p+q+r \Rightarrow -\frac{u}{\lambda} = 1 \Rightarrow \lambda = -u$$

$$x = -\frac{pu}{\lambda a}=\frac{-pu}{-ua}=\frac{p}{a}$$

$$y = -\frac{qu}{\lambda b}=\frac{-qu}{-ub}=\frac{q}{b}$$

$$z = -\frac{ru}{\lambda c}=\frac{-ru}{-uc}=\frac{r}{c}$$

Putting in (1), we have

$$\text{Maximum value of } u = \left(\frac{p}{a}\right)^p \left(\frac{q}{b}\right)^q \left(\frac{r}{c}\right)^r \qquad \textbf{Ans.}$$

Home Work

Question 1: A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring the least material for its construction.

Question 2: Divide 24 into three parts such that the continued product of the first part, the square of the second part, and the cube of the third part is maximum.

Answer 1: $l = 4$, $b = 4$ and $h = 2$ units

Answer 2: $x = 12$, $y = 8$, $z = 4$

cont.

3. Decompose a positive number 'a' into three parts so that their product is maximum.

Ans. $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}$

4. The sum, of three numbers is constant. Prove that their product is a maximum when they are equal.

5. Using the method of Lagrange's multipliers, find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

Ans. $\frac{4096}{25\sqrt{5}}$

LECTURE 23

PROBLEMS ON LAGRANGE'S METHOD OF MULTIPLIERS

QUESTION – 1

Find the minimum distance from the point (1, 2, 0) to the cone $z^2 = x^2 + y^2$.

Sol. Let (x, y, z) be any point on the cone then distance from the point (1, 2, 0) is

$$D^2 = (x-1)^2 + (y-2)^2 + (z-0)^2$$

Let $$u = (x-1)^2 + (y-2)^2 + z^2 \quad \ldots(i)$$

Subject to $$x^2 + y^2 - z^2 = 0 \quad \ldots(ii)$$

for minimum, from (i) and (ii), we get

$$du = (x-1)dx + (y-2)dy + zdz = 0 \quad \ldots(iii)$$

$$xdx + ydy - zdz = 0 \quad \ldots(iv)$$

Multiplying equation (iv) by λ and adding in (iii), we get

$$(x-1)dx + (y-2)dy + zdz + \lambda\,(xdx + ydy - zdz) = 0$$

$$\Rightarrow \quad \{x(1+\lambda) - 1\}\,dx + \{y(1+\lambda) - 2\}\,dy + \{z(1-\lambda)\}\,dz = 0$$

$$\Rightarrow \quad x(1+\lambda) - 1 = 0,\ y(1+\lambda) - 2 = 0,\ z(1-\lambda) = 0$$

$$\Rightarrow \quad x = \frac{1}{1+\lambda},\quad y = \frac{2}{1+\lambda},\quad \lambda = 1 \quad \ldots(v)$$

$$\therefore \qquad x = \frac{1}{1+1} = \frac{1}{2}, \quad y = \frac{2}{1+1} = 1$$

Putting the value of x and y in equation (ii), we get

$$\frac{1}{4} + 1 - z^2 = 0 \Rightarrow z^2 = \frac{5}{4} \Rightarrow z = \pm \frac{\sqrt{5}}{2}$$

Hence, the minimum distance from the point (1, 2, 0) is

$$D^2 = \left(\frac{1}{2}-1\right)^2 + (1-2)^2 + \left(\frac{\sqrt{5}}{2}\right)^2 = \frac{1}{4}+1+\frac{5}{4} = \frac{10}{4}$$

$$D^2 = \frac{5}{2} \Rightarrow D = \sqrt{\frac{5}{2}}.$$

QUESTION – 2

Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol. Let the length, breadth and height of solid are

$$l = 2x$$

$$b = 2y$$

$$h = 2z$$

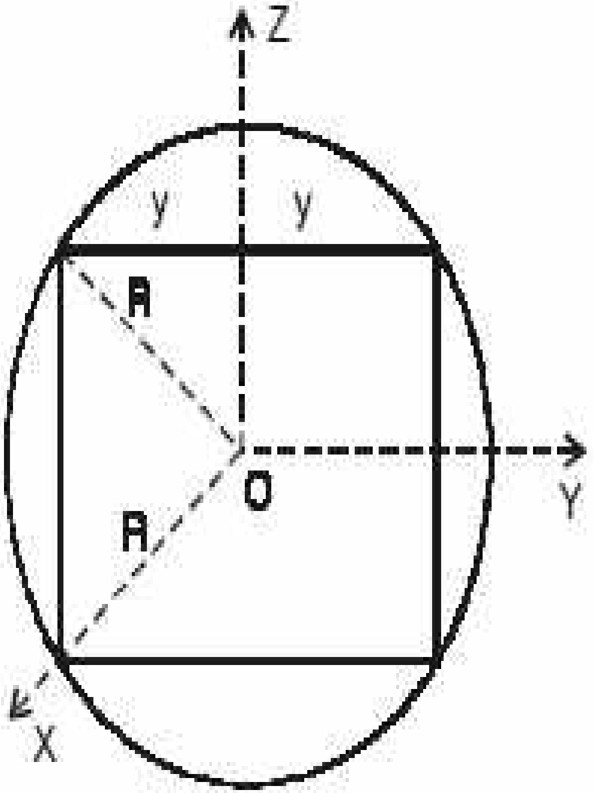

Fig. 2.8

$\therefore$ Volume of the solid $V = lbh = 2x \cdot 2y \cdot 2z$

$\Rightarrow \quad V = 8xyz \quad$ ...(*i*)

Equation of the sphere

$$x^2 + y^2 + z^2 = R^2$$

$\Rightarrow \quad x^2 + y^2 + z^2 - R^2 = 0 \quad$ ...(*ii*)

For maximum differentiating (*i*), (*ii*), we get

$dV = 8yzdx + 8xzdy + 8xydz = 0 \quad$ ...(*iii*)

$2xdx + 2ydy + 2zdz = 0 \quad$ ...(*iv*)

Multiplying (*iv*) by λ and adding in (*iii*), we get

$$8yzdx + 8xzdy + 8xydz + \lambda\,(2xdx + 2ydy + 2zdz) = 0$$

$$\Rightarrow \quad (2\lambda x + 8yz)dx + (2\lambda y + 8xz)dy + (2\lambda z + 8xy)dz = 0$$

Equating the coefficient of dx, dy and dz to zero, we get

$$\Rightarrow \quad \lambda x = -4yz\,,\ \lambda y = -4xz,\ \lambda z = -4xy \qquad \ldots(v)$$

These are Lagrange's equations

Multiplying equation (*v*) by x, y, z respectively, we get

$$\lambda x^2 = -4xyz,\ \lambda y^2 = -4xyz,\ \lambda z^2 = -4xyz$$

From these, we get

$$\lambda x^2 = \lambda y^2 = \lambda z^2$$

$$\Rightarrow \quad x^2 = y^2 = z^2$$

$$\Rightarrow \quad x = y = z$$

Thus, the rectangular solid is a cube. **Proved.**

Example

Find the dimension of rectangular box of maximum capacity whose surface area is given when

(*a*) box is open at the top (*b*) box is closed.

Sol. Let the length, breadth and height of box are x, y, z respectively.

So volume $V = xyz$ ...(*i*)

There will be two surface area one for open and one for closed box

$\therefore \quad nxy + 2yz + 2zx = S$ (say) ...(*ii*)

or $\quad g(x, y, z) \equiv nxy + 2yz + 2zx - S = 0$ ...(*iii*)

Here $n = 1$, when the box is open on the top
$n = 2$, when the box is closed.

The Lagrange's equations are

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial g}{\partial x} = yz + \lambda(ny + 2z) = 0 \quad ...(iv)$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial g}{\partial y} = xz + \lambda(nx + 2z) = 0 \quad ...(v)$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial g}{\partial z} = xy + \lambda(2y + 2x) = 0 \quad ...(vi)$$

Multiplying (*iv*), (*v*), (*vi*) by x, y, z respectively and adding, we get

$$3xyz + \lambda\,[2(nxy + 2yz + 2zx)] = 0$$

or $$3V + \lambda[2S] = 0 \Rightarrow \lambda = -\frac{3V}{2S} \quad \text{...(vii)}$$

Putting value of λ from (*vii*) in (*iv*), (*v*) and (*vi*), we get

$$yz - \frac{3V}{2S}(ny + 2z) = 0 \Rightarrow yz - \frac{3xyz}{2S}(ny + 2z) = 0$$

or $$nxy + 2xz = \frac{2S}{3} \quad \text{...(viii)}$$

Similarly $$nxy + 2yz = \frac{2S}{3} \quad \text{...(ix)}$$

$$2yz + 2zx = \frac{2S}{3} \quad \text{...(x)}$$

From (*viii*) and (*ix*), we get

$$x = y \quad \text{...(xi)}$$

and from (*ix*), (*x*), we get

$$ny = 2z \Rightarrow z = \frac{ny}{2} = \frac{nx}{2} \quad \text{...(xii)}$$

Putting (*xi*) and (*xii*) in equation (*ii*), we have

$$nx \cdot x + 2 \cdot x \cdot \frac{nx}{2} + 2 \cdot \frac{nx}{2} \cdot x = S \quad \Rightarrow \quad 3nx^2 = S$$

or

$$x^2 = \frac{S}{3n}$$

(*a*) When box is open $n = 1$

$$\therefore \quad x^2 = \frac{S}{3} \quad \Rightarrow \quad x = \sqrt{\frac{S}{3}}$$

Hence, the dimensions of the open box are $x = y = \sqrt{\frac{S}{3}}$ and $z = \frac{1}{2}\sqrt{\frac{S}{3}}$

(*b*) When box is closed $n = 2 \quad \therefore \quad x^2 = \frac{S}{6} \quad \Rightarrow \quad x = \sqrt{\frac{S}{6}}$

Hence, the dimensions of the closed box are

$$x = y = \sqrt{\frac{S}{6}} \quad \text{and} \quad z = \sqrt{\frac{S}{6}}.$$

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Q.4 Find the maximum and minimum distance of the point $(1, 2, -1)$ from the sphere $x^2 + y^2 + z^2 = 24$. (2017-18) (2018-19)

Sol. Let (x, y, z) be any point on the sphere.

Distance of the point $A(1, 2, -1)$ from (x, y, z) is given by $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2$ — (1)

Subject to the condition that $\phi(x, y, z) \equiv x^2 + y^2 + z^2 - 24 = 0$ — (2)

Consider Lagrange's function

$$F(x,y,z) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-24)$$

For stationary point $dF = 0$

$$\Rightarrow [2(x-1)+2\lambda x]\,dx + [2(y-2)+2\lambda y]\,dy + [2(z+1)+2\lambda z]\,dz = 0$$

$\Rightarrow 2(x-1)+2\lambda x = 0$ or $(x-1)+\lambda x = 0$ — (3)

$2(y-2)+2\lambda y = 0$ or $(y-2)+\lambda y = 0$ — (4)

$2(z+1)+2\lambda z = 0$ or $(z+1)+\lambda z = 0$ — (5)

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$(x^2+y^2+z^2) - x - 2y + z + \lambda(x^2+y^2+z^2) = 0$$

or $x+2y-z = 24(1+\lambda)$ — (6) (using (2))

From (3), (4) and (5), $x = \frac{1}{1+\lambda}$, $y = \frac{2}{1+\lambda}$, $z = \frac{-1}{1+\lambda}$

Putting these values of x, y, z in (6), we get

$$\frac{1}{1+\lambda} + \frac{4}{1+\lambda} + \frac{1}{1+\lambda} = 24(1+\lambda)$$

or $(1+\lambda)^2 = \frac{1}{4}$ or $1+\lambda = \pm\frac{1}{2}$

$\therefore 1+\lambda = \frac{1}{2} \Rightarrow x = 2; y = 4, z = -2$; Point $P(2, 4, -2)$

$1+\lambda = -\frac{1}{2} \Rightarrow x = -2, y = -4, z = 2$; Point $Q(-2, -4, 2)$

Now $AP = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2} = \sqrt{1+4+1} = \sqrt{6}$

and $AQ = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2} = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$

$\therefore$ P(2, 4, −2) is at a minimum distance from A and the minimum distance $= \sqrt{6}$

Q(−2, −4, 2) is at a maximum distance from A and the maximum distance $= 3\sqrt{6}$.

Q5. Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$. (2011-12)

Sol. Let (x, y) be any point on the curve. Distance of the point $A(0,0)$ from (x, y) is given by $\sqrt{x^2+y^2}$

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x,y) = x^2 + y^2$ —— ①

Subject to the condition

$$\phi(x,y) \equiv x^2 + 4xy + 6y^2 - 140 = 0 \quad \text{—— ②}$$

Consider Lagrange's function

$$F(x,y) = f(x,y) + \lambda \phi(x,y)$$
$$= x^2+y^2+\lambda\,(x^2+4xy+6y^2-140)$$

For stationary values $dF=0 \Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$

$\Rightarrow [2x+\lambda(2x+4y)]dx + [2y+\lambda(4x+12y)]\,dy = 0$

$\Rightarrow \quad 2x+\lambda(2x+4y)=0 \quad$ or $\quad x+\lambda(x+2y)=0$ — (3)

and $\quad 2y+\lambda(4x+12y)=0 \quad$ or $\quad y+\lambda(2x+6y)=0$ — (4)

Multiplying eq (3) by x, (4) by y and adding

$$x^2+y^2+\lambda(x^2+4xy+6y^2)=0$$

$\Rightarrow \quad f+\lambda(140)=0 \quad \Rightarrow \lambda = -\frac{f}{140}$, using (1) & (2)

eq (3) $\Rightarrow$ $x - \frac{f}{140}(x+2y) = 0$ $\Rightarrow (140-f)x - 2fy = 0$ —— (5)

eq (4) $\rightarrow$ $y - \frac{f}{140}(2x+6y) = 0 \Rightarrow -fx + (70-3f)y = 0$ —— (6)

Solving (5) and (6), we get

$$f^2 - 490f + 9800 = 0$$

$$\Rightarrow f = \frac{490 \pm \sqrt{(490)^2 - 4(9800)}}{2} = 245 \pm 35\sqrt{41}$$

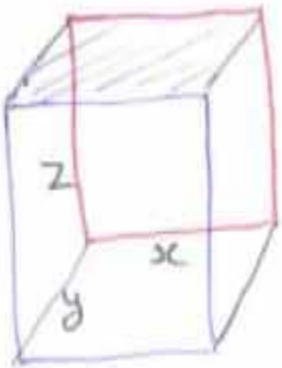
$$= 469.1093 \; , \; 20.8906$$

Hence maximum distance $= \sqrt{469.1093} = 21.6589$

and minimum distance $= \sqrt{20.8906} = 4.5706$

Q6 A rectangular box, open at the top is to have a capacity of 32 c.c. Determine, using Lagrange's method of multipliers, the dimensions of the box such that the least material is required for the construction of the box. (2014-15)

Sol. Let x, y, z be the dimensions of the rectangular box, open at the top.



Volume $V = xyz = 32$ (given)

Surface area of the box is

Surface area of the box is

$$S = xy + 2yz + 2zx$$ (which is to be minimized)

Consider Lagrange's function

$$F(x,y,z) = xy + 2yz + 2zx + \lambda(xyz - 32)$$

For stationary point $dF = 0$

$$\Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

$$\Rightarrow (y + 2z + \lambda yz)dx + (x + 2z + \lambda xz)dy + (2y + 2x + \lambda xy)dz = 0$$

$\Rightarrow y + 2z + \lambda yz = 0$ — (1)

$x + 2z + \lambda xz = 0$ — (2)

$2y + 2x + \lambda xy = 0$ — (3)

Multiplying eq. (1), (2), (3) by x, y, z respectively, we get

$xy + 2xz + \lambda xyz = 0$ — (4)

$xy + 2yz + \lambda xyz = 0$ — (5)

$2yz + 2zx + \lambda xyz = 0$ — (6)

eq (4) − eq (5) ⇒ $2xz - 2yz = 0 \Rightarrow 2z(x-y) = 0 \Rightarrow x = y$

eq (5) − eq (6) ⇒ $xy - 2zx = 0 \Rightarrow x(y-2z) = 0 \Rightarrow y = 2z$

Hence $x = y = 2z$

i.e. Length = breadth = 2 height

Now given $xyz = 32$

$\Rightarrow (2z)(2z)z = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2$

Hence $x = y = 4$, $z = 2$. Ans.

Example . *A rectangular box, which is open at the top, has a capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is required for the construction of the box. Use Lagrange's method of multipliers to obtain the solution.*

Solution. Let x, y, z be the length, breadth and height of the box.

$$\Rightarrow \quad \text{Volume} = xyz = 256 \quad \Rightarrow \quad xyz - 256 = 0 \quad \quad ...(1)$$

$$\Rightarrow \quad \phi(x, y, z) = x\,y\,z - 256$$

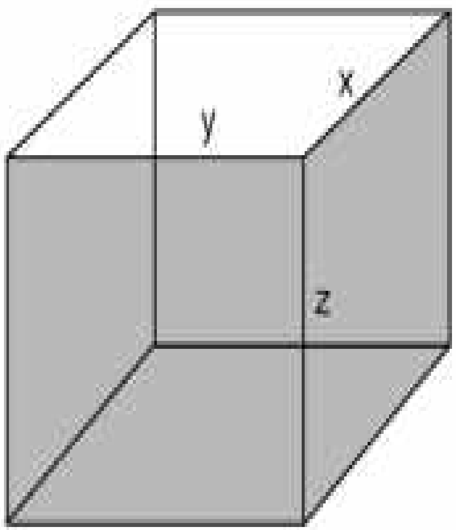
Let S be the material surface of the box.

$$S = x\,y + 2\,y\,z + 2\,z\,x$$

$$\frac{\partial S}{\partial x} = y + 2z \quad \text{and} \quad \frac{\partial \phi}{\partial x} = y\,z$$

$$\frac{\partial S}{\partial y} = x + 2z \quad \text{and} \quad \frac{\partial \phi}{\partial y} = x\,z$$

$$\frac{\partial S}{\partial z} = 2y + 2x \quad \text{and} \quad \frac{\partial \phi}{\partial z} = x\,y$$



By Lagrange's method of multiplier, we have

$$\frac{\partial S}{\partial x}+\lambda\frac{\partial \phi}{\partial x} = 0 \quad \Rightarrow \quad y + 2z + \lambda\, yz = 0 \quad \text{...(2)}$$

$$\frac{\partial S}{\partial y}+\lambda\frac{\partial \phi}{\partial y} = 0 \quad \Rightarrow \quad x + 2z + \lambda\, xz = 0 \quad \text{...(3)}$$

$$\frac{\partial S}{\partial z}+\lambda\frac{\partial \phi}{\partial z} = 0 \quad \Rightarrow \quad 2y + 2x + \lambda\, xy = 0 \quad \text{...(4)}$$

Multiplying (2) by x, we get

$$xy + 2\,xz + \lambda\, xyz = 0$$

$$\Rightarrow \quad xy + 2\,xz + 256\,\lambda = 0 \qquad (xyz = 256)$$

$$\Rightarrow \quad xy + 2\,xz = -256\,\lambda \quad \text{...(5)}$$

Multiplying (3) by y, we get

$$xy + 2\,yz + \lambda\, xyz = 0$$

$$\Rightarrow \quad xy + 2\,yz + 256\,\lambda = 0$$

$$\Rightarrow \quad xy + 2\,yz = -256\,\lambda \quad \text{...(6)}$$

Multiplying (4) by z, we get

$$2\,yz + 2\,xz + \lambda\, xyz = 0 \quad \Rightarrow \quad 2\,yz + 2\,xz + 256\,\lambda = 0$$

$$\Rightarrow \quad 2\,yz + 2\,zx = -256\,\lambda \quad \text{...(7)}$$

From (5) and (6), we have

$$xy + 2\,xz = xy + 2\,yz \quad \Rightarrow \quad 2\,xz = 2\,yz \;\Rightarrow\; x = y$$

From (6) and (7), we have

$$xy + 2\,yz = 2\,yz + 2\,xz \quad \Rightarrow \quad xy = 2\,xz \;\Rightarrow\; y = 2z$$

From (1) $\qquad xyz = 256$

$$\Rightarrow \qquad (y)\,(y)\left(\frac{y}{2}\right) = 256 \;\Rightarrow\; y^3 = 512 \;\Rightarrow\; y = 8$$

$$x = 8,\quad y = 8,\quad z = 4$$

Hence, length = breadth = 8', height = 4'. **Ans.**

HOME WORK

1. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere, $x^2 + y^2 + z^2 = 1$. (*U.P.T.U., 2001*) $\left[\textbf{Ans. } D_{min} = 12,\ D_{max} = 14\right]$

2. Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$. [*U.P.T.U. (C.O.), 2003*] $\left[\textbf{Ans. } D_{min} = 4.5706,\ D_{max} = 21.6589\right]$

3. The temperature T at any point (x, y, z) in space is $T = 400\ xyz^2$. Find the highest temperature at the surface of a sphere $x^2 + y^2 + z^2 = 1$. $\left[\textbf{Ans. } T = 50\right]$

4. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions : $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = 0$.

$$\left[\textbf{Ans. } \frac{l^2}{(au-1)} + \frac{m^2}{(bu-1)} + \frac{n^2}{(cu-1)} = 0\right]$$

5. Find the maximum value of $u = x^p y^q z^r$ when the variables x, y, z are subject to the condition $ax + by + cz = p + q + r$.

$$\left[\textbf{Ans. } u = \left(\frac{p}{a}\right)^p \left(\frac{q}{b}\right)^q \left(\frac{r}{c}\right)^r\right]$$

LECTURE 24

Introduction to Jacobian

JACOBIAN

The Jacobians themselves are of great importance in transformation of variables from one coordinate system to another coordinate system (Cartesian to polar etc.). They are also useful in area and volume for surface and volume integrals.

DEFINITION

If $u = u(x, y)$ and $v = v(x, y)$ where x and y are independent, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is known as the Jacobian of u, v with respect to x, y and is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J(u, v)$$

Similarly, the Jacobian of three functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ is defined as

$$J(u, v, w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

Q1 If $u = x(1-y)$, $v = xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$ (2019-20)

Sol. $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$

$= x(1-y) + xy = x$

Q2. If $x = e^v \sec u$, $y = e^v \tan u$ then evaluate $\frac{\partial(x,y)}{\partial(u,v)}$

(2020-21)

Sol.

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix}$

$$= e^{2v} \sec u \tan^2 u - e^{2v} \sec^3 u$$

$$= e^{2v} \sec u (\tan^2 u - \sec^2 u) \qquad (\because 1 + \tan^2 u = \sec^2 u)$$

$$= e^{2v} \sec u (-1) = -e^{2v} \sec u$$

Q3. If $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_1 x_3}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$, then find the value of $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$ (2017 - 18)

OR

If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, Show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

Solution: Here given

$$y_1 = \frac{x_2 x_3}{x_1}, \quad y_2 = \frac{x_3 x_1}{x_2}, \quad y_3 = \frac{x_1 x_2}{x_3}$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_2} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1\ (1-1)\ -1(-1-1) + 1(1+1)$

$= 0 + 2 + 2$

$= 4$ Proved

Example 4.

If $u = x + y + z$, $uv = y + z$, $u\,v\,w = z$, Evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Solution:

$x = u - uv = u\,(1 - v)$

$y = uv - uvw = uv\,(1 - w)$

$z = uvw$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uw \\ vw & uw & uv \end{vmatrix}$$

Applying $R_1 \to R_1 + (R_2 + R_3)$

$= u^2v\ (1 - w) + u^2\ vw$

$= u^2v$ Answer.

HOME WORK

Q 1. If $x = r\cos\theta$, $y = r\sin\theta$ find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$. $\left[\text{Ans. } \dfrac{1}{r}\right]$

Q 2 If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$$

Q 3. If $u = \dfrac{y^2}{2x}$, $v = \dfrac{(x^2+y^2)}{2x}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. $\left.\text{Ans. } -\dfrac{y}{2x}\right]$

LECTURE 24

Properties of Jacobian

1. If $u = u(x, y)$ and $v = v(x, y)$, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1 \text{ or } JJ' = 1$$

2. Chain rule: If u, v, are function of r, s and r, s are themselves functions of x, y *i.e.*,

$u = u(r, s)$, $v = v(r, s)$ and $r = r(x, y)$, $s = s(x, y)$

then
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

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Q1 If $x = u(1+v)$, $y = v(1+u)$ then find the Jacobian of u, v with respect to x, y) (2017-18)

Sol

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv = 1+u+v$$

By chain rules $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{1+u+v}$

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Q. If $x = r\cos\theta$, $y = r\sin\theta$, $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ then show that $JJ^* = 1$

Or

Verify the chain rule for Jacobians if $x = r\cos\theta$, $y = r\sin\theta$.

Sol $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r$$

Now $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

Hence $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1 \Rightarrow J\,J^* = 1$

Thus chain rule is verified.

Example 3. (2014-2015)

If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r \sin\theta \cos\phi$, $v = r \sin\theta \sin\phi$, $w = r\cos\theta$, calculate $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Sol. Here *x, y, z are functions of u, v, w and u, v, w are functions* r, θ, ϕ ***so we apply the* property.**

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} \qquad \ldots(i)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\left[\sqrt{\frac{w}{v}\frac{v}{u}\frac{u}{w}} + \sqrt{\frac{v}{w}\frac{w}{u}\frac{u}{v}}\right] = \frac{1}{8}\left[\sqrt{1}+\sqrt{1}\right] = \frac{2}{8} = \frac{1}{4}$$

Next $$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)}$$

$$= \sin\theta \cos\phi\,(r^2 \sin^2\theta \cos\phi) - r\cos\theta\cos\phi\,(-r^2 \sin\theta\cos\theta\cos\phi)$$
$$+ r^2 \sin\theta \sin\phi\,(\sin^2\theta \sin\phi + \cos^2\theta \sin\phi)$$

$$= r^2 \sin\theta \cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r^2 \sin\theta \sin^2\phi$$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2 \sin\theta \cos^2\phi + r^2 \sin\theta \sin^2\phi = r^2 \sin\theta \qquad \ldots(iii)$$

Using (*ii*) and (*iii*) in equation (*i*), we get

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{1}{4} \times r^2 \sin\theta = \frac{r^2 \sin\theta}{4}.$$

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Q4. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ then find

$$\frac{\partial(r, \theta, z)}{\partial(x, y, z)}$$

(2018-19)

Sol

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

By chain rule

$$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(r,\theta,z)}} = \frac{1}{r}$$

Q5. Calculate $\frac{\partial(u,v)}{\partial(x,y)}$ for $x = e^u \cos v$ and $y = e^u \sin v$

(2011-12)

Sol. Proceed as above

$(\text{Ans} = e^{-2u})$

Example 5. Verify the chain rule for Jacobians if $x = u$, $y = u \tan v$, $z = w$.

Sol. We have

$$x = u \quad \Rightarrow \quad \frac{\partial x}{\partial u} = 1, \ \frac{\partial x}{\partial v} = \frac{\partial x}{\partial w} = 0$$

$$y = u \tan v \quad \Rightarrow \quad \frac{\partial y}{\partial u} = \tan v, \ \frac{\partial y}{\partial v} = u \sec^2 v, \ \frac{\partial y}{\partial w} = 0$$

$$z = w \quad \Rightarrow \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial v} = 0, \ \frac{\partial z}{\partial w} = 1$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u \sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v \qquad \ldots(i)$$

Solving for u, v, w in terms of x, y, z, we have

$$u = x$$

$$v = \tan^{-1} \frac{y}{u} = \tan^{-1} \frac{y}{x}$$

$$w = z$$

$$\therefore \quad \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}, \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2},$$

$$\frac{\partial v}{\partial z} = 0, \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \text{ and } \frac{\partial w}{\partial z} = 1$$

$$J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2+y^2} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{1}{u \sec^2 v} \quad \ldots(ii)$$

Hence from (*i*) and (*ii*), we get

$$J.J' = u \sec^2 v . \frac{1}{u \sec^2 v} = 1.$$

Example 6. If $u = x\,(1 - r^2)^{-1/2}$, $v = y\,(1 - r^2)^{-1/2}$, $w = z\,(1 - r^2)^{-1/2}$

where $r = \sqrt{x^2 + y^2 + z^2}$, then find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

Sol. Since $\quad r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Differentiating partially $u = x\,(1 - r^2)^{-1/2}$ w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \left(1-r^2\right)^{\frac{-1}{2}} + x\left(\frac{-1}{2}\right)(-2r)\left(1-r^2\right)^{\frac{-3}{2}} \cdot \frac{\partial r}{\partial x}$$

$$= \left(1-r^2\right)^{\frac{-1}{2}} + rx\left(1-r^2\right)^{\frac{-3}{2}} \cdot \frac{x}{r} = \frac{1}{\sqrt{1-r^2}} + \frac{x^2}{\left(1-r^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1-r^2+x^2}{\left(1-r^2\right)^{\frac{3}{2}}}$$

Differentiating partially u w.r.t. y, we get

$$\frac{\partial u}{\partial y} = x\left(\frac{-1}{2}\right)\left(1-r^2\right)^{\frac{-3}{2}} \cdot (-2r)\frac{\partial r}{\partial y} = \frac{xr}{\left(1-r^2\right)^{\frac{3}{2}}} \cdot \frac{y}{r}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial y} = \frac{xy}{\left(1-r^2\right)^{\frac{3}{2}}}$$

and

$$\frac{\partial u}{\partial z} = \frac{xz}{\left(1-r^2\right)^{\frac{3}{2}}}$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{yx}{\left(1-r^2\right)^{\frac{3}{2}}}, \quad \frac{\partial v}{\partial y} = \frac{1-r^2+y^2}{\left(1-r^2\right)^{\frac{3}{2}}}, \quad \frac{\partial v}{\partial z} = \frac{yz}{\left(1-r^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial w}{\partial x} = \frac{zx}{\left(1-r^2\right)^{\frac{3}{2}}}, \quad \frac{\partial w}{\partial y} = \frac{zy}{\left(1-r^2\right)^{\frac{3}{2}}}, \quad \frac{\partial w}{\partial z} = \frac{1-r^2+z^2}{\left(1-r^2\right)^{\frac{3}{2}}}$$

Thus, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{1-r^2+x^2}{(1-r^2)^{\frac{3}{2}}} & \frac{xy}{(1-r^2)^{\frac{3}{2}}} & \frac{xz}{(1-r^2)^{\frac{3}{2}}} \\ \frac{yx}{(1-r^2)^{\frac{3}{2}}} & \frac{1-r^2+y^2}{(1-r^2)^{\frac{3}{2}}} & \frac{yz}{(1-r^2)^{\frac{3}{2}}} \\ \frac{zx}{(1-r^2)^{\frac{3}{2}}} & \frac{zy}{(1-r^2)^{\frac{3}{2}}} & \frac{1-r^2+z^2}{(1-r^2)^{\frac{3}{2}}} \end{vmatrix}$$

$$= \frac{1}{(1-r^2)^{\frac{9}{2}}} \begin{vmatrix} 1-r^2+x^2 & xy & xz \\ yx & 1-r^2+y^2 & yz \\ zx & zy & 1-r^2+z^2 \end{vmatrix}$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1 - r^2 + x^2) \{(1 - r^2 + y^2)(1 - r^2 + z^2) - y^2z^2\}$$

$$- xy \{xy (1 - r^2 + z^2) - xyz^2\} + xz\{xy^2z - zx(1 - r^2 + y^2)\}]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1 - r^2 + x^2) (1 - r^2 + y^2) (1 - r^2 + z^2)$$

$$- (1 - r^2) (y^2z^2 + x^2y^2 + x^2z^2) - x^2y^2z^2]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1 - r^2)^3 + (1 - r^2)^2 (x^2 + y^2 + z^2)]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1 - r^2)^3 + (1 - r^2)^2 r^2]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} . (1 - r^2)^2 [1 - r^2 + r^2] = \left(1-r^2\right)^{\frac{-5}{2}}.$$

HOME WORK

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Q 1. Calculate $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ and $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$. Verify that $JJ' = 1$ given

(i) $u = x + \dfrac{y^2}{x}$, $v = \dfrac{y^2}{x}$. $\left[\text{Ans. } J = \dfrac{2y}{x}, J' = \dfrac{x}{2y}\right]$

(ii) $x = e^u \cos v$, $y = e^u \sin v$. $\left[\text{Ans. } J = e^{2u}, J' = e^{-2u}\right]$

Q 2. Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$ given $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

LECTURE 25

Jacobian of Implicit Functions

JACOBIAN OF IMPLICIT FUNCTIONS

If the variables u, v and x, y be connected by the equations

$$f_1(u, v, x, y) = 0$$
$$f_2(u, v, x, y) = 0$$

i.e., u, v are implicit functions of x, y.

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

Similarly for three variables u, v, w

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

Example 1.

If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}$.

Sol. Let

$$f_1 \equiv u^3 + v^3 - x - y = 0$$

$$f_2 \equiv u^2 + v^2 - x^3 - y^3 = 0$$

Now,

$$\frac{\partial(f_1, f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} = 3(y^2 - x^2)$$

and

$$\frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6\,uv\,(u - v)$$

Thus,

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\frac{\partial(f_1, f_2)}{\partial(x,y)}}{\frac{\partial(f_1, f_2)}{\partial(u,v)}} = \frac{3(y^2 - x^2)}{6uv(u-v)} = \frac{(y^2 - x^2)}{2uv(u-v)}.$$

Example 2. (2018-2019)

u, v, w are the roots of the equation

$(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$, find $\dfrac{\partial(u, v, w)}{\partial(a, b, c)}$.

Sol. We have $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$

$x^3 - a^3 - 3xa(x-a) + x^3 - b^3 - 3xb(x-b) + x^3 - c^3 - 3xc(x-c) = 0$

or $3x^3 - 3x^2(a+b+c) + 3x(a^2+b^2+c^2) - (a^3+b^3+c^3) = 0$

Since u, v, w are the roots of this equation, we have

$$u + v + w = a + b + c$$

$$uv + vw + wu = a^2 + b^2 + c^2$$

$$uvw = \frac{a^3 + b^3 + c^3}{3}$$

IF $Ax^3 + Bx^2 + Cx + D = 0$

Then, sum of roots $= \alpha + \beta + \gamma = -B/A = \dfrac{-(\text{coefficient of } x^2)}{\text{coefficient of } x^3}$

Sum of product of roots taken, two at a time

$$\alpha\beta + \beta\gamma + \alpha\gamma = C/A = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of roots $= -D/A \Rightarrow \alpha\beta\gamma = \dfrac{-(\text{constant term})}{\text{coefficient of } x^3}$

Let

$$f_1 \equiv u + v + w - a - b - c = 0$$

$$f_2 \equiv uv + vw + wu - a^2 - b^2 - c^2 = 0$$

$$f_3 = uvw - \frac{a^3 + b^3 + c^3}{3}$$

Now

$$\frac{\partial(f_1, f_2, f_3)}{\partial(a, b, c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -2a & 2(a-b) & 2(a-c) \\ -a^2 & (a^2-b^2) & (a^2-c^2) \end{vmatrix}, \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix}$$

$$= -2\{(a-b)(a^2-c^2) - (a-c)(a^2-b^2)\} = -2(a-b)(b-c)(c-a)$$

and

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & v+u \\ vw & wu & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}, \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix}$$

$$= (u-v)\,v(u-w) - (u-w)\,w(u-v)$$

$$= -(u-v)(v-w)(w-u)$$

Thus

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{-2(a-b)(b-c)(c-a)}{-(u-v)(v-w)(w-u)}$$

$$= -2\frac{(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}.$$

Q3. If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ then find

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}$$

(2015-16)

Sol. Proceed as in last question.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Example 4. (2019-2020)

If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$,

$u + v + w = x^2 + y^2 + z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

Sol. Let
$$\begin{aligned} f_1 &\equiv u^3 + v^3 + w^3 - x - y - z = 0 \\ f_2 &\equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0 \\ f_3 &\equiv u + v + w - x^2 - y^2 - z^2 = 0 \end{aligned}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -3x^2 & 3(x^2-y^2) & 3(x^2-z^2) \\ -2x & 2(x-y) & 2(x-z) \end{vmatrix} \quad |C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$$

$$= -6[(x^2-y^2)(x-z) - (x^2-z^2)(x-y)]$$

$$= -6(x-y)(x-z)[(x+y)-(x+z)]$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = 6(x-y)(y-z)(z-x)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3u^2 & 3(v^2-u^2) & 3(w^2-u^2) \\ 2u & 2(v-u) & 2(w-u) \\ 1 & 0 & 0 \end{vmatrix} \quad \Big| \, C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$$

Expand it with respect to third row, we get

$$= 6[(v^2-u^2)(w-u) - (w^2-u^2)(v-u)]$$

$$= 6(v-u)(w-u)[(v+u)-(w+u)]$$

$$\Rightarrow \quad \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = -6(u-v)(v-w)(w-u)$$

Hence $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = +\frac{6(x-y)(y-z)(z-x)}{6(u-v)(v-w)(w-u)}$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$ Hence proved.

Ques. If $u^3+v+w = x+y^2+z^2$

$$u+v^3+w = x^2+y+z^2$$

$$u+v+w^3 = x^2+y^2+z$$

Show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4xy(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$$

(2020-21)

Hint: Proceed as above

Functional Dependence

Let $u = f_1(x, y)$, $v = f_2(x, y)$ be two functions. Suppose u and v are connected by the relation $f(u, v) = 0$, where f is differentiable. Then u and v are called functionally dependent on one another

Hence, two functions u and v are "functionally dependent" if their Jacobian is equal to zero.

Note: The functions u and v are said to be "functionally independent" if their Jacobian is not equal to zero *i.e.*, $J(u, v) \neq 0$

Similarly for three functionally dependent functions say u, v and w.

$$J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$$

Q(1) Are the functions $u = \frac{x-y}{x+z}$, $v = \frac{x+z}{y+z}$ functionally dependent? If so, find the relation between them. (2011-12)

Sol. Note that here u, v are taking functions of any two variables (say x and z), therefore z is treating as a constant.

Now

$$\frac{\partial u}{\partial x} = \frac{(x+z)-(x-y)}{(x+z)^2} = \frac{y+z}{(x+z)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+z}(0-1) = -\frac{1}{x+z}$$

$$\frac{\partial v}{\partial x} = \frac{1}{y+z}(1+0) = \frac{1}{y+z}$$

$$\frac{\partial v}{\partial y} = \frac{-1}{(y+z)^2}(x+z) = \frac{-(x+z)}{(y+z)^2}$$

Now

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{-(y+z)}{(x+z)^2}\,\frac{(x+z)}{(y+z)^2} + \frac{1}{(x+z)}\cdot\frac{1}{(y+z)}$$

$$= -\frac{1}{(x+z)(y+z)} + \frac{1}{(x+z)(y+z)} = 0$$

Hence, the functional relationship exists between

Now $1-u = 1 - \frac{x-y}{x+z} = \frac{x+z-x+y}{x+z} = \frac{y+z}{x+z} = \frac{1}{v}$

Hence $\boxed{1-u = \frac{1}{v} \quad \text{or} \quad \frac{1}{1-u} = v}$

Q. If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$

Show that they are not independent.

Find the relation between u, v and w. (2016-17)

Sol.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y - z & 2x - 4y + 6z & -x + 2y - 3z \end{vmatrix}$$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - C_1$

$$= -4(-x+2y-3z) - 2(2x-4y+6z)$$

$$= 4x - 8y + 12z - 4x + 8y - 12z$$

$$= 0$$

Hence u, v, w are not independent.

Now $u+v = 2x+4z$

$u-v = 4y-2z$

Multiplying these, we get

$$(u+v)(u-v) = (2x+4z)(4y-2z)$$

$$= 4(2xy+4yz-2x-2z^2)$$

$$\Rightarrow \boxed{u^2 - v^2 = 4w}$$

Q3. Show that $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ are not independent. Find the relation between them. (2013-14)

Sol. Relation is $2u^2 = v - w$

Hint: $J(u, v, w) = 0$ shows u, v, w are not independent.

Q4. Show that the functions

$u = x + y + z$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$

$w = x^3 + y^3 + z^3 - 3xyz$

are functionally related. Find the relation between them.

(2013-14, 2017-18)

Sol $J(u, v, w) = 0$ shows u, v, w are dependent and relation is $w = \dfrac{u(u^2 + 3v)}{4}$

Proceed as above.

Example 5.

Show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between them.

Sol. Here $u = x + y - z,\ v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2yz$

Now,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2y-2z & 0 \end{vmatrix} \quad (C_3 \to C_3 + C_2)$$

$= 0$. Hence u, v, w are not independent.

$$
\begin{aligned}
\text{Again} \qquad u + v &= x + y - z + x - y + z = 2x \\
u - v &= x + y - z - x + y - z = 2(y - z) \\
\therefore \quad (u + v)^2 + (u - v)^2 &= 4x^2 + 4(y - z)^2 \\
&= 4(x^2 + y^2 + z^2 - 2yz) = 4w \\
\Rightarrow \quad (u + v)^2 + (u - v)^2 &= 4w \\
\text{or} \qquad 2(u^2 + v^2) &= 4w \text{ or } u^2 + v^2 = 2w.
\end{aligned}
$$

Example 6.

Find Jacobian of $u = \sin^{-1}x + \sin^{-1}y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$.
Also find the relation between them.

Sol. We have $u = \sin^{-1}x + \sin^{-1}y,\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Now,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= -\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} = 0.$$ Hence u and v are dependent.

$$u = \sin^{-1}x + \sin^{-1}y \Rightarrow u = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$$

$$\left| \text{As } \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left\{A\sqrt{1-B^2} + B\sqrt{1-A^2}\right\}\right.$$

$$\sin u = x\sqrt{1-y^2} + y\sqrt{1-x^2} = v$$

$$\Rightarrow \quad v = \sin u.$$

HOME WORK

Q 1. If $u = x + y + z,\ uv = y + z,\ uvw = z$, show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v.$

Q 2. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{x^2 - y^2}{u^2 + v^2}.$

Q 3. If $u = x_1 + x_2 + x_3 + x_4,\ uv = x_2 + x_3 + x_4,\ uvw = x_3 + x_4$ and $uvwt = x_4$, show that

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u, v, w, t)} = u^3v^2w.$$

HOME WORK

Q 4. If $u = \dfrac{x}{y-z}, v = \dfrac{y}{z-x}, w = \dfrac{z}{x-y}$, then prove that u, v, w are not independent and also find the relation between them. $[\text{Ans. } uv+vw+wu+1=0]$

Q 5. If $u = x + 2y + z, v = x - 2y + 3z, w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w. $[\text{Ans. } 4w = u^2 - v^2]$

Lecture -26
Approximation of Errors

Approximation of Errors

If δx and δy are small change (or errors) in x and y respectively, then an approximate change (or error) in z is δz.

Note- ① If δx is the error in x, then

② relative error $= \dfrac{\delta x}{x}$

③ percentage error $= \dfrac{\delta x}{x} \times 100$.

Q.1. The period of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$.

Find the maximum (%) error in T due to the possible errors upto 1% in l and 2.5% in g.

(2013-14)

Sol. $T = 2\pi\sqrt{\frac{l}{g}}$

Taking log, $\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$

Differentiating, we get

$$\frac{1}{T}\delta T = \frac{1}{2}\frac{\delta l}{l} - \frac{1}{2}\frac{\delta g}{g}$$

$$\Rightarrow \frac{\delta T}{T}\times 100 = \frac{1}{2}\left[\frac{\delta l}{l}\times 100 - \frac{\delta g}{g}\times 100\right]$$

$$= \frac{1}{2}(1 \pm 2.5)$$

Given

$$\frac{\delta l}{l}\times 100 = 1$$

$$\frac{\delta g}{g}\times 100 = 2.5$$

Maximum error $= \frac{1}{2}(1 + 2.5)$

$= 1.75\%$

Q8. What error in the common logarithm of a number will be produced by an error of 1% in the number? (2017-18)

Sol. Let x as any number

and $y = \log_{10} x$ $\therefore y = \log_e x \cdot \log_{10} e$

Then $\delta y = \frac{1}{x} \log_{10} e \, (\delta x)$

$= \left(\frac{\delta x}{x} \times 100\right)\left(\frac{1}{100} \log_{10} e\right)$ $\left(\text{Given } \frac{\delta x}{x} \times 100 = 1\right)$

$= \frac{1}{100} (0.43429)$

$= 0.0043429$

Years (2013-2014, 2017-2018)

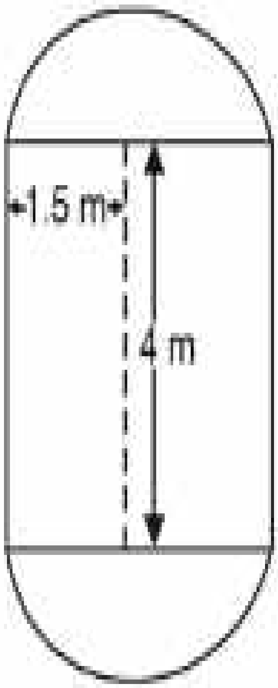
Example . *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the percentage change in the volume of the balloon.*

(U.P. I Sem., Dec., 2005, Comp 2002)

Solution. Radius of the cylinder (r) = 1.5 m

Length of the cylinder (h) = 4 m

Volume of the balloon = Volume of cylinder + Volume of two hemispheres



$$\text{Volume } (V) = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\delta V = \pi\, 2r\, \delta r . h + \pi r^2 . \delta h + \frac{4}{3}\pi\, 3r^2 . \delta r$$

$$\frac{\delta V}{V} = \frac{\pi r [2\,\delta r . h + r . \delta h + 4 r \delta r]}{\pi r^2 h + \frac{4}{3} \pi r^3} = \frac{2 . \delta r . h + r . \delta h + 4 r . \delta r}{r h + \frac{4}{3} r^2}$$

$$= \frac{2 \times 0.01 \times 4 + 1.5 \times 0.05 + 4 \times 1.5 \times 0.01}{1.5 \times 4 + \frac{4}{3} (1.5)^2}$$

$$= \frac{0.08 + 0.075 + 0.06}{6 + 3} = \frac{0.215}{9}$$

$$100 \frac{\delta V}{V} = \frac{100 \times 0.215}{9} = \frac{21.5}{9} = 2.389\%$$

Ans.

Q4 Find approximate value of $[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}}$

(2011-12
2013-14)

Sol. Let $f(x,y) = (x^2 + 2y^3)^{\frac{1}{5}}$

Taking $x = 4$, $\delta x = 3.82 - 4 = -0.18$

$y = 2$, $\delta y = 2.1 - 2 = 0.1$

$$\frac{\partial f}{\partial x} = \frac{1}{5}(x^2+2y^3)^{-\frac{4}{5}}.(2x) = \frac{2}{5}(4)(16+2(2)^3)^{-\frac{4}{5}} = \frac{8}{5}(32)^{-\frac{4}{5}}$$

$$= \frac{8}{5}(2^5)^{-\frac{4}{5}} = \frac{8}{5} 2^{-4} = \frac{8}{5}\cdot\frac{1}{2^4} = \frac{8}{5}\cdot\frac{1}{16} = \frac{1}{10}$$

$$\frac{\partial f}{\partial y} = \frac{1}{5}(x^2+2y^3)^{-\frac{4}{5}}(6y^2) = \frac{6}{5}(4)[16+2(2)^3]^{-\frac{4}{5}} = \frac{24}{5}\left(\frac{1}{16}\right) = \frac{3}{10}$$

$$df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$= \frac{1}{10}(-0.18) + \frac{3}{10}(0.1) = -0.018 + 0.03 = 0.012$$

Hence

$$[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}} = f(4,2) + df$$

$$= [(4)^2 + 2(2)^3]^{\frac{1}{5}} + 0.012$$

$$= 2 + 0.012 = 2.012$$

Q.5 Find approximate value of

$$[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2}$$

Sol. Let $f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$

Taking $x = 1$, $y = 2$, $z = 2$, so that

$\delta x = -0.02$, $\delta y = 0.01$, $\delta z = -0.06$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x = \frac{x}{(x^2+y^2+z^2)^{1/2}}$$

Similarly $\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{1/2}}$, $\frac{\partial f}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}}$

Now $df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}}(x\,\delta x + y\,\delta y + z\,\delta z)$$

$$= \frac{1}{(1+4+4)^{1/2}}(-0.02 + 0.02 - 0.12)$$

$$= \frac{1}{3}(-0.12) = -0.04$$

$$\therefore [(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2} = f(1,2,2) + df$$

$$= (1+4+4)^{1/2} - 0.04 = 3 - 0.04 = 2.96$$

HOME WORK

1 The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $s^2 D^{2/3} t^2$.

Find approximately the percentage increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance is increased by 3%. **Ans.** $\frac{14}{3}\%$

2 The power P required to propel a ship of length l moving with a velocity V is given by $P = kV^3 l^2$. Find the percentage increase in power if increase in velocity is 3% and increase in length is 4%.

Ans. 17%

3 In estimating the cost of a pile of bricks measured as 2m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length if the count is 450 bricks to 1 m^3 and bricks cost ₹ 1300 per 1000, find the approximate error in the cost. **Ans.** ₹ 631.80

4 In estimating the cost of a pile of bricks measured as 6′ × 50′ × 4′, the tape is stretched 1% beyond the standard length. If the count is 12 bricks to ft^3, and bricks cost ₹ 100 per 1000, find the approximate error in the cost. *(U.P. I Sem., Dec. 2004)* **Ans.** 720 bricks, ₹ 25.20

Lecture -26
Approximation of Errors

Example The power dissipated in a resistor is given by $P = \frac{E^2}{R}$. Find by using calculus the approximate percentage change in P when E is increased by 3% and R is decreased by 2%.

Solution: Here given $P = \frac{E^2}{R}$

Taking logarithm we have

$$\log P = 2 \log E - \log R$$

on differentiating, we get

$$\frac{\delta P}{P} = \frac{2}{E}\delta E - \frac{\delta R}{R}$$

$$\text{or } 100\frac{\delta P}{P} = 2 \times \frac{100\delta E}{E} - \frac{100\delta R}{R}$$

$$\text{or } 100\frac{\delta P}{P} = 2(3) - (-2)$$

$$= 8$$

Percentage change in P = 8% Answer.

Example *In estimating the number of bricks in a pile which is measured to be (5m × 10m × 5m), count of bricks is taken as 100 bricks per m³. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is ₹ 2,000 per thousand bricks.* *(U.P., I Semester, Winter 2000)*

Solution. Volume $V = x\,y\,z$

$$\log V = \log x + \log y + \log z$$

Differentiating, we get

$$\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$100\frac{\delta V}{V} = \frac{100\,\delta x}{x} + \frac{100\,\delta y}{y} + \frac{100\,\delta z}{z} = 2 + 2 + 2$$

$$\frac{100\,\delta V}{V} = 6$$

$$\delta V = \frac{6V}{100} = \frac{6\,(5 \times 10 \times 5)}{100} = 15 \text{ cubicmetre.}$$

Number of bricks in $\delta V = 15 \times 100 = 1500$

$$\text{Error in cost} = \frac{1500 \times 2000}{1000} = 3000$$

Thus error in cost, a loss to the seller of bricks = ₹ 3000. **Ans.**

GROUP OF INSTITUTIONS

Example. If $f(x, y) = x^2 y^{\frac{1}{10}}$, compute the value of f when $x = 1.99$ and $y = 3.01$.

Sol. We have $f(x, y) = x^2 y^{\frac{1}{10}}$

$$\therefore \quad \frac{\partial f}{\partial x} = 2xy^{\frac{1}{10}}, \quad \frac{\partial f}{\partial y} = \frac{1}{10} x^2 y^{-\frac{9}{10}}$$

Let $x = 2, \delta x = -0.01$

$y = 1, \delta y = 2.01$

As $x + \delta x = 2 + (-0.01) = 1.99$
$y + \delta y = 1 + (2.01) = 3.01$

Now, $f(x + \delta x, y + \delta y) = f(x, y) + \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$

$$\Rightarrow f\{2 + (-0.01), 1 + 2.01\} = f(2, 1) + 2 \times 2(1)^{\frac{1}{10}} \times (-0.01) + \frac{1}{10} (2)^2 . (1)^{-\frac{9}{10}} \times (2.01)$$

$$\Rightarrow \quad f(1.99, 3.01) \approx 2^2 \times 1^{\frac{1}{10}} + (-0.04) + 0.804$$

$$\approx 4 - 0.04 + 0.804 = 4.764.$$

HOME WORK

1. A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m find the percentage change in the volume of the balloon. **Ans 2.389%**

2. The formula, $V = kr^4$, says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V? Ans 40%

3. Find approximate value of : $\left[(3.82)^2 + 2(2.1)^3\right]^{1/5}$ **Ans 2.012**