

Unit- 5

Vector Calculus

Lecture 37(I)

Gradient

Scalar Point Function

If to each point $P(x, y, z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function.

For example, the temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

Mathematically

$f(x, y, z) = x^2 + 2yz^5$ is an example of scalar function

Vector Point Function

If to each point $P(x, y, z)$ of a region R in space there corresponds a unique vector $f(P)$, then f is called a *vector point function*.

The velocity of a moving fluid, gravitational force are the examples of vector point function.

Mathematically

$r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$, be an example of vector point function

Vector Differential Operator

The vector differential operator Del is denoted by ∇ . It is defined as

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

GRADIENT OF SCALAR FIELD

If $\phi(x, y, z)$ be a scalar function then $\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ .

And is denoted by $\text{grad } \phi$.

Thus,

$$\text{grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\text{grad } \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi(x, y, z)$$

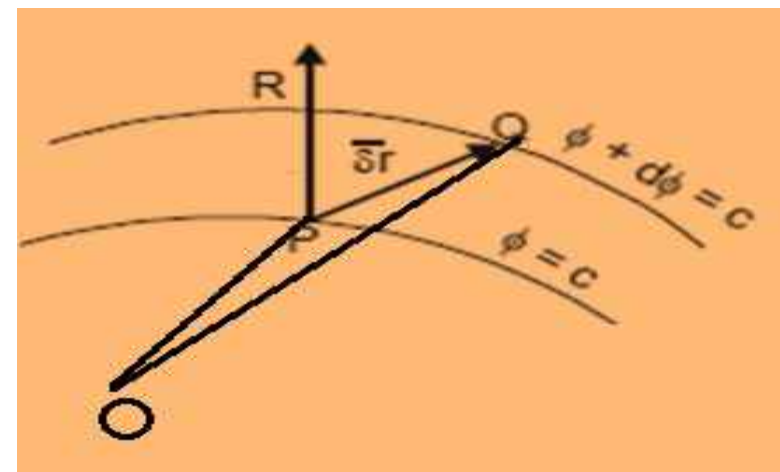
$$\text{grad } \phi = \nabla \phi$$

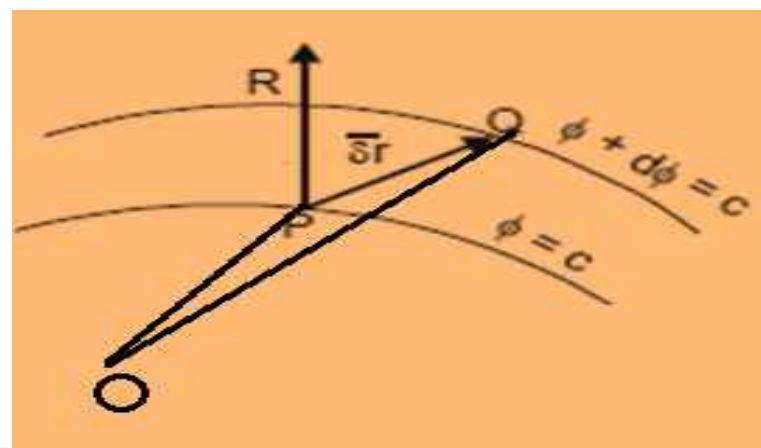
GEOMETRICAL INTERPRETATION

If a surface $\phi(x, y, z) = c$ passes through a point P . The value of the function at each point on the surface is the same as at P . Then such a surface is called a *level surface* through P . For example, If $\phi(x, y, z)$ represents potential at the point P , then *equipotential surface* $\phi(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ . Consider another level surface passing through Q , where the value of the function is $\phi + d\phi$.





Let \vec{r} and $\vec{r} + \delta\vec{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\vec{r}$

$$\begin{aligned} \nabla\phi \cdot d\vec{r} &= \left(\hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z} \right) \cdot (\hat{i} dx + \hat{j} dy + \hat{k} dz) \\ &= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi \end{aligned} \quad \dots(1)$$

If Q lies on the level surface of P , then $d\phi = 0$

Equation (1) becomes $\nabla\phi \cdot d\vec{r} = 0$. Then $\nabla\phi$ is \perp to $d\vec{r}$ (tangent).

Hence, $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$

PROPERTIES OF GRADIENT

(a) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(b) If ϕ_1 and ϕ_2 are two scalar point functions, then

$$(i) \nabla (\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$$

(ii) $\nabla (c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constant

$$(iii) \nabla (\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$$

$$(iv) \nabla \left(\frac{\phi_1}{\phi_2} \right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}, \phi_2 \neq 0.$$

EXAMPLE 1:

Find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point $(1, -2, -1)$.

SOLUTION:

$$\begin{aligned}\text{Grad } \phi &= \nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (3x^2y - y^3z^2) \\ &= \hat{i} \frac{\partial}{\partial x} (3x^2y - y^3z^2) + \hat{j} \frac{\partial}{\partial y} (3x^2y - y^3z^2) + \hat{k} \frac{\partial}{\partial z} (3x^2y - y^3z^2) \\ &= \hat{i} (6xy) + \hat{j} (3x^2 - 3y^2z^2) + \hat{k} (-2y^3z) \\ &= -12\hat{i} - 9\hat{j} - 16\hat{k} \text{ at the point } (1, -2, -1).\end{aligned}$$

EXAMPLE 2.

Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point

$(1, 2, -1)$.

SOLUTION

Let $\phi = x^3 + y^3 + 3xyz - 3$, then $\frac{\partial \phi}{\partial x} = 3x^2 + 3yz$, $\frac{\partial \phi}{\partial y} = 3y^2 + 3xz$, $\frac{\partial \phi}{\partial z} = 3xy$

$$\nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

$$\text{At } (1, 2, -1), \nabla\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$$

which is a vector normal to the given surface at $(1, 2, -1)$.

Hence a unit vector normal to the given surface at $(1, 2, -1)$

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{(-3)^2 + (9)^2 + (6)^2}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

EXAMPLE 3

If $\nabla\phi = (y^2 - 2xyz^3) i + (3 + 2xy - x^2z^3) j + (6z^3 - 3x^2yz^2) k$, find ϕ .

SOLUTION

$$\text{Let } \vec{F} = \nabla\phi$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r}$$

$$= \left(\frac{\partial\phi}{\partial x} \hat{i} + \frac{\partial\phi}{\partial y} \hat{j} + \frac{\partial\phi}{\partial z} \hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$$

$$\therefore d\phi = \vec{F} \cdot d\vec{r}$$

$$= (y^2 - 2xyz^3) \hat{i} + (3 + 2xy - x^2z^3) \hat{j} + (6z^3 - 3x^2yz^2) \hat{k} \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= (y^2 - 2xyz^3) dx + (3 + 2xy - x^2z^3) dy + (6z^3 - 3x^2yz^2) dz$$

$$= (y^2 dx + 2xy dy) - (2xyz^3 dx + x^2z^3 dy + 3x^2yz^2 dz) + 3 dy + 6z^3 dz$$

$$= d(xy^2) - d(x^2yz^3) + d(3y) + d\left(\frac{3}{2}z^4\right)$$

$$\therefore \phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + c$$

EXAMPLE 4

If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that

$$(\text{grad } u) \cdot [(\text{grad } v) \times (\text{grad } w)] = 0.$$

SOLUTION

$$\begin{aligned} \text{We have grad } u &= \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} + \frac{\partial u}{\partial z} \mathbf{k} \\ &= 1 \mathbf{i} + 1 \mathbf{j} + 1 \mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \end{aligned}$$

$$\text{grad } v = \frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} + \frac{\partial v}{\partial z} \mathbf{k} = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}$$

$$\text{and grad } w = \frac{\partial w}{\partial x} \mathbf{i} + \frac{\partial w}{\partial y} \mathbf{j} + \frac{\partial w}{\partial z} \mathbf{k}$$

$\therefore \text{grad } u \cdot [(\text{grad } v) \times (\text{grad } w)] = \text{scalar triple product of the vectors grad } u, \text{ grad } v \text{ and grad } w$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} \text{ by } R_2 + R_3$$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2(x+y+z) \cdot 0 = 0$$

EXAMPLE 5

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that

$$(i) \text{grad } r = \frac{\vec{r}}{r}$$

$$(ii) \text{grad} \left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^3}$$

$$(iii) \nabla r^n = nr^{n-2} \vec{r}$$

SOLUTION

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, \quad \text{or} \quad r^2 = x^2 + y^2 + z^2$$

Differentiating partially w.r.t. x , we have $2r \frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

(i)

$$\text{grad } r = \nabla r = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) r = \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z}$$

$$= \hat{i} \left(\frac{x}{r} \right) + \hat{j} \left(\frac{y}{r} \right) + \hat{k} \left(\frac{z}{r} \right) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}$$

(ii)

$$\text{grad} \left(\frac{1}{r} \right) = \nabla \left(\frac{1}{r} \right) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(\frac{1}{r} \right)$$

$$= \hat{i} \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) + \hat{j} \left(-\frac{1}{r^2} \frac{\partial r}{\partial y} \right) + \hat{k} \left(-\frac{1}{r^2} \frac{\partial r}{\partial z} \right)$$

$$= \hat{i} \left(-\frac{1}{r^2} \cdot \frac{x}{r} \right) + \hat{j} \left(-\frac{1}{r^2} \cdot \frac{y}{r} \right) + \hat{k} \left(-\frac{1}{r^2} \cdot \frac{z}{r} \right)$$

$$= -\frac{1}{r^3} (x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}$$

(iii)

$$\begin{aligned}\nabla r^n &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) r^n = \hat{i} \left(nr^{n-1} \frac{\partial r}{\partial x} \right) + \hat{j} \left(nr^{n-1} \frac{\partial r}{\partial y} \right) + \hat{k} \left(nr^{n-1} \frac{\partial r}{\partial z} \right) \\ &= \hat{i} \left(nr^{n-1} \cdot \frac{x}{r} \right) + \hat{j} \left(nr^{n-1} \cdot \frac{y}{r} \right) + \hat{k} \left(nr^{n-1} \cdot \frac{z}{r} \right) = nr^{n-2} (x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2} \vec{r}.\end{aligned}$$

EXAMPLE 6

Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

SOLUTION

Let $\phi_1 = x^2 + y^2 + z^2 = 9$ and $\phi_2 = x^2 + y^2 - z = 3$

Then $\text{grad } \phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and $\text{grad } \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n}_1 = \text{grad } \phi_1$ at the point $(2, -1, 2)$ and $\vec{n}_2 = \text{grad } \phi_2$ at the point $(2, -1, 2)$. Then

$$\vec{n}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \text{and} \quad \vec{n}_2 = 4\hat{i} - 2\hat{j} - \hat{k}$$

The vectors \vec{n}_1 and \vec{n}_2 are along normals to the two surfaces at the point $(2, -1, 2)$. If θ is the angle between these vectors, then

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16 + 4 + 16} \cdot \sqrt{16 + 4 + 1}} = \frac{16}{6\sqrt{21}}$$

\therefore

$$\theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right).$$

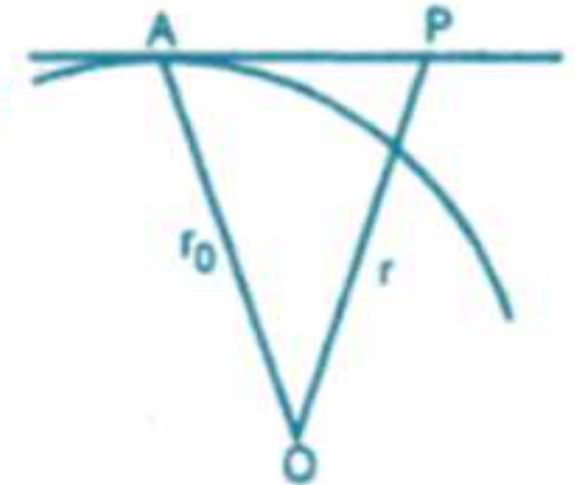
APPLICATION OF GRADIENT EQUATION OF THE TANGENT PLANE

Tangent Plane. Let \mathbf{r}_0 , be the position vector of the point of contact A and \mathbf{r} be the position vector of any point P on the tangent plane.

Then $\mathbf{r} - \mathbf{r}_0$ is a vector parallel to the tangent plane and $\text{grad } f$ is normal to the tangent plane. These two are perpendicular

$$\therefore (\mathbf{r} - \mathbf{r}_0) \cdot \text{grad } f = 0$$

(1)



This equation will be satisfied by any point \mathbf{r} lying in the tangent plane. Moreover, for any point with position vector \mathbf{r} which satisfies (1) the vector $(\mathbf{r} - \mathbf{r}_0)$ is parallel to the tangent plane. It follows that $\mathbf{r} - \mathbf{r}_0$ lies in the plane; hence the end point of \mathbf{r} is in the plane. Therefore (1) is the equation of the tangent plane.

EXAMPLE 7

Find the equation of the tangent plane and normal to the surface $xyz = 3$ at the point $(1, 2, 2)$.

SOLUTION

$$\text{grad } f = yzi + zxj + xyk = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Also

$$\begin{aligned}\mathbf{r} - \mathbf{r}_0 &= (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ &= (x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 2)\mathbf{k}.\end{aligned}$$

The equation to the tangent plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \text{grad } f = 0$$

$$\therefore \{(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 2)\mathbf{k}\} \cdot (4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 0$$

$$\text{i.e., } 4(x - 1) + 2(y - 2) + 2(z - 2) = 0$$

$$\text{i.e., } 2x + y + z = 6.$$

Gradient in Polar Co-ordinate

If $f(r)$ is a scalar function of scalar r then it's gradient is given by

$$\text{grad } f = \frac{\partial}{\partial r} (f) \cdot \hat{r}$$

$$r = |\vec{r}| \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

Ex: 1

Prove that $\text{grad } r^n = n r^{n-2} \vec{r}$ where $r = |\vec{r}|$

Sol: 1

$$\text{grad } r^n = \frac{\partial}{\partial r} (r^n) \cdot \hat{r} = n r^{n-1} \hat{r} = n r^{n-2} \vec{r}$$

where \hat{r} is the unit vector in the direction of \vec{r} .

Practice Questions

Q1

If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

$$(i) \nabla f(r) = f'(r) \nabla r$$

$$(ii) \nabla \log r = \frac{\vec{r}}{r^2}$$

Q2

If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$, show that

$$\cos \theta = \frac{3}{7\sqrt{6}}$$

Practice Questions

Q3 Evaluate grad ϕ if $\phi = \log (x^2 + y^2 + z^2)$

Ans. $\frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$

Q4 Find a unit normal vector to the surface $z^2 = x^2 + y^2$ at the point $(1, 0, -1)$.

Ans. $\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

Lecture 37(II)

Directional Derivatives

Directional Derivatives

Directional Derivative

Let $f(x, y, z)$ be a scalar valued function, directional derivative of $f(x, y, z)$ at the point \vec{a} in the direction of a vector \vec{b} is given by

$$\text{Directional derivative} = (\nabla f)_{at \vec{b}} \cdot \hat{a}$$

\hat{a} is the unit vector of the vector \vec{a} .

Example 1

What is the directional derivative of the function $xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$?

Solution

$$\phi(x, y, z) = xy^2 + yz^3$$

$$\begin{aligned}\text{Gradient of } \phi &= \nabla\phi = \hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z} \\ &= \hat{i} y^2 + \hat{j} (2xy + z^3) + \hat{k} (3yz^2)\end{aligned}$$

$$\nabla\phi \text{ at } (2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$$

If \hat{n} is a unit vector in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$, then $\hat{n} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

∴ Directional derivative of the given function ϕ at $(2, -1, 1)$ in the direction of

$$\hat{i} + 2\hat{j} + 2\hat{k} = [\nabla\phi \text{ at } (2, -1, 1)] \cdot \hat{n}$$

$$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{1-6-6}{3} = -\frac{11}{3}$$

Example 2

Find the directional derivative of $\phi(x, y, z) = x^2 y z + 4 x z^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find the greatest rate of increase of ϕ .

Solution

Here, $\phi(x, y, z) = x^2 y z + 4 x z^2$

Now,
$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 y z + 4 x z^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2 z)\hat{j} + (x^2 y + 8xz)\hat{k}$$

$$\nabla\phi \text{ at } (1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\hat{i} + (1 \times 1)\hat{j} + \{1(-2) + 8(1)(1)\}\hat{k}$$

$$= (-4 + 4)\hat{i} + \hat{j} + (-2 + 8)\hat{k} = \hat{j} + 6\hat{k}$$

Let
$$\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

Let $\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$

So, the required directional derivative at $(1, -2, 1)$

$$= \nabla\phi \cdot \hat{a} = (\hat{j} + 6\hat{k}) \cdot \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{1}{3}(-1 - 12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = \left| \hat{j} + 6\hat{k} \right| = \sqrt{1+36}$
 $= \sqrt{37}$

Example 3

Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$.

In what direction will it be maximum? Find also the magnitude of this maximum.

Solution

$$\text{We have } \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} = 2\hat{i} - 4\hat{j} + 12\hat{k} \text{ at } P(1, 2, 3)$$

$$\text{Also } \vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\text{If } \hat{n} \text{ is a unit vector in the direction } \vec{PQ}, \text{ then } \hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{1}{\sqrt{21}} (4\hat{i} - 2\hat{j} + \hat{k})$$

$$\begin{aligned}\therefore \text{ Directional derivative of } f \text{ in the direction } \vec{PQ} &= (\nabla f) \cdot \hat{n} \\ &= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}} (4\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{21}} [2(4) - 4(-2) + 12(1)] \\ &= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}\end{aligned}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

$$\begin{aligned}\text{The maximum value of this directional derivative} &= |\nabla f| \\ &= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}.\end{aligned}$$

Example 4

Find the directional derivative of $\phi = e^{2x} \cos yz$ at the origin in the direction of the tangent to the curve $x = a \sin t$, $y = a \cos t$, $z = at$ at $t = \frac{\pi}{4}$.

Solution

$$\begin{aligned} \text{Gradient of } \phi = \nabla \phi &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (e^{2x} \cos yz) \\ &= \hat{i} (2e^{2x} \cos yz) + \hat{j} (-e^{2x} z \sin yz) + \hat{k} \{ e^{2x} (-\sin yz) y \} \end{aligned}$$

At the origin i.e., when $x = 0$, $y = 0$, $z = 0$,

$$\nabla \phi = \hat{i} (2) = 2\hat{i}$$

Equation of the curve is $x = a \sin t, y = a \cos t, z = at$

Any point on the curve is $\vec{r} = \hat{i} (a \sin t) + \hat{j} (a \cos t) + \hat{k} (at)$

Direction of the tangent is given by $= \frac{d\vec{r}}{dt} = (a \cos t) \hat{i} - (a \sin t) \hat{j} + a \hat{k}$

At $t = \frac{\pi}{4}$, direction of tangent $= \frac{a}{\sqrt{2}} \hat{i} - \frac{a}{\sqrt{2}} \hat{j} + a \hat{k}$

\hat{n} = unit direction of the tangent

$$= \frac{\frac{a}{\sqrt{2}} \hat{i} - \frac{a}{\sqrt{2}} \hat{j} + a \hat{k}}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{\frac{a}{\sqrt{2}} (\hat{i} - \hat{j} + \sqrt{2} \hat{k})}{\sqrt{2} a} = \frac{1}{2} (\hat{i} - \hat{j} + \sqrt{2} \hat{k})$$

Directional derivative of ϕ at $(0, 0, 0)$ in the direction of tangent at $t = \frac{\pi}{4}$ is $= \nabla \phi \cdot \hat{n}$ at

$$(0, 0, 0). \quad = 2\hat{i} \cdot \frac{1}{2} (\hat{i} - \hat{j} + \sqrt{2}\hat{k}) = 1.$$

Example 5

Find the directional derivative of $\nabla \cdot (\nabla f)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.

Solution

Here, we have

$$f = 2x^3y^2z^4$$

$$\nabla f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (2x^3y^2z^4) = 6x^2y^2z^4 \hat{i} + 4x^3yz^4 \hat{j} + 8x^3y^2z^3 \hat{k}$$

$$\begin{aligned} \nabla \cdot (\nabla f) &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (6x^2y^2z^4 \hat{i} + 4x^3yz^4 \hat{j} + 8x^3y^2z^3 \hat{k}) \\ &= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \end{aligned}$$

Gradient of $\nabla \cdot (\nabla f)$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz^4 + 48x^3yz^2)\hat{j} + (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

Gradient of $\nabla \cdot (\nabla f)$ at $(1, -2, 1) = (48 + 12 + 288)\hat{i} + (-48 - 96)\hat{j} + (192 + 16 + 192)\hat{k}$

$$= 348\hat{i} - 144\hat{j} + 400\hat{k}$$

Normal to $(xy^2z - 3x - z^2) = \nabla(xy^2z - 3x - z^2)$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

Normal at $(1, -2, 1) = \hat{i} - 4\hat{j} + 2\hat{k}$

$$\text{Unit Normal Vector} = \frac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1 + 16 + 4}} = \frac{1}{\sqrt{21}} (\hat{i} - 4\hat{j} + 2\hat{k})$$

Directional derivative in the direction of normal

$$= (348\hat{i} - 144\hat{j} + 400\hat{k}) \cdot \frac{1}{\sqrt{21}} (\hat{i} - 4\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{21}} (348 + 576 + 800) = \frac{1724}{\sqrt{21}}$$

Q.3: → Find the directional derivative of v^2 , where $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point $(2, 0, 3)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(3, 2, 1)$.

Solⁿ: → $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ then $v^2 = x^2y^4 + z^2y^4 + x^2z^4 \equiv f$ (2012-13)

Now

$$\begin{aligned}\nabla f &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2y^4 + z^2y^4 + x^2z^4) \\ &= \hat{i} (2xy^4 + 2xz^4) + \hat{j} (4x^2y^3 + 4z^2y^3) + \hat{k} (2zy^4 + 4x^2z^3)\end{aligned}$$

At point $(2, 0, 3)$, $\nabla f = 324\hat{i} + 432\hat{k}$

Normal to the sphere $x^2 + y^2 + z^2 = 14 \equiv \phi$ is

$$\nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 14) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At $(3, 2, 1)$ $\nabla \phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Normal to the sphere $x^2+y^2+z^2=14 \equiv \phi$ is

$$\nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2+y^2+z^2-14) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At $(3, 2, 1)$ $\nabla \phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

If \hat{n} is a unit vector in outward normal to the sphere

then $\hat{n} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36+16+4}} = \frac{1}{\sqrt{56}} (6\hat{i} + 4\hat{j} + 2\hat{k})$

\therefore Directional derivative of f in the outward normal to the

sphere = $\nabla f \cdot \hat{n}$

$$= (324\hat{i} + 432\hat{k}) \cdot \frac{1}{\sqrt{56}} (6\hat{i} + 4\hat{j} + 2\hat{k}) = \frac{1944 + 864}{\sqrt{56}} = \frac{1404}{\sqrt{14}}$$

Q-2: Find the directional derivative of $\phi = (x^2 + y^2 + z^2)^{-1/2}$ at the point $(3, 1, 2)$ in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$.

Solⁿ: $\phi = (x^2 + y^2 + z^2)^{-1/2}$

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-1/2}$$

$$= \hat{i} \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x) \right\} + \hat{j} \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2y) \right\}$$

$$+ \hat{k} \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2z) \right\}$$

$$= - \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}} = - \frac{3\hat{i} + \hat{j} + 2\hat{k}}{14\sqrt{14}} \text{ at } (3, 1, 2)$$

Let \hat{a} be the unit vector in the given direction, then

$$\hat{a} = \frac{yz\hat{i} + zx\hat{j} + xy\hat{k}}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}} = \frac{2\hat{i} + 6\hat{j} + 3\hat{k}}{7} \text{ at } (3, 1, 2)$$

$$\begin{aligned}
 \therefore \text{Directional derivative} &= \hat{a} \cdot \nabla \phi \\
 &= \frac{2\hat{i} + 6\hat{j} + 3\hat{k}}{7} \cdot \left(-\frac{3\hat{i} + \hat{j} + 2\hat{k}}{14\sqrt{14}} \right) \\
 &= -\frac{6 + 6 + 6}{7 \cdot 14\sqrt{14}} = -\frac{9}{49\sqrt{14}}.
 \end{aligned}$$

Q.3: Find the directional derivative of $\left(\frac{1}{x^2}\right)$ in the direction of \vec{a} , where $\vec{a} = \hat{i}x + \hat{j}y + \hat{k}z$. (2016-17)

Solⁿ: $\nabla\left(\frac{1}{x^2}\right) = \frac{2}{x^3} \hat{i} = -\frac{2}{x^4} \cdot \vec{a}$ ($\because \hat{i} = \frac{\vec{a}}{|\vec{a}|}$)

Let \hat{a} be the unit vector in the direction of \vec{a} then $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$$\therefore \text{Directional derivative} = \nabla\left(\frac{1}{x^2}\right) \cdot \hat{a} = -\frac{2}{x^4} \cdot \vec{a} \cdot \frac{\vec{a}}{|\vec{a}|} = -\frac{2}{x^3}$$

Q.41: Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

(2018-19)

Soln: $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$$\therefore \text{grad } \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(5x^2y - 5y^2z + \frac{5}{2}z^2x \right)$$

$$= \left(10xy + \frac{5}{2}z^2 \right) \hat{i} + (5x^2 - 10yz) \hat{j} + (-5y^2 + 5zx) \hat{k}$$

$$= \frac{25}{2} \hat{i} - 5 \hat{j} \quad \text{at } (1,1,1)$$

Here $\hat{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$

$$\therefore \text{Directional derivative} = (\text{grad } \phi) \cdot \hat{a}$$

$$= \left(\frac{25}{2} \hat{i} - 5 \hat{j} \right) \cdot \left(\frac{2}{3} \hat{i} - \frac{2}{3} \hat{j} + \frac{1}{3} \hat{k} \right)$$

$$= \frac{25}{3} + \frac{10}{3} = \frac{35}{3}$$

Q.5: Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find also the greatest rate of increase of ϕ . (2019-20)

Solⁿ: $\phi(x, y, z) = x^2yz + 4xz^2$

$$\therefore \nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2yz + 4xz^2)$$
$$= \hat{i} (2xyz + 4z^2) + \hat{j} (x^2z) + \hat{k} (x^2y + 8xz)$$

At $(1, -2, 1)$, $\nabla \phi = \hat{j} + 6\hat{k}$

If \hat{n} is a unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$, then

$$\hat{n} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3} (2\hat{i} - \hat{j} - 2\hat{k})$$

So the required directional derivative at $(1, -2, 1)$

$$= \nabla \phi \cdot \hat{n} = (\hat{j} + 6\hat{k}) \cdot \frac{1}{3} (2\hat{i} - \hat{j} - 2\hat{k}) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = |\hat{j} + 6\hat{k}| = \sqrt{1+36} = \sqrt{37}$

Q.3: → Find the directional derivative of v^2 , where $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point $(2, 0, 3)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(3, 2, 1)$.

Solⁿ: → $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ then $v^2 = x^2y^4 + z^2y^4 + x^2z^4 \equiv f$ (2012-13)

Now

$$\begin{aligned}\nabla f &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2y^4 + z^2y^4 + x^2z^4) \\ &= \hat{i} (2xy^4 + 2xz^4) + \hat{j} (4x^2y^3 + 4z^2y^3) + \hat{k} (2zy^4 + 4x^2z^3)\end{aligned}$$

At point $(2, 0, 3)$, $\nabla f = 324\hat{i} + 432\hat{k}$

Normal to the sphere $x^2 + y^2 + z^2 = 14 \equiv \phi$ is

$$\nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 14) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At $(3, 2, 1)$ $\nabla \phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Normal to the sphere $x^2 + y^2 + z^2 = 14 \equiv \phi$ is

$$\nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 14) = 2x \hat{i} + 2y \hat{j} + 2z \hat{k}$$

At $(3, 2, 1)$ $\nabla \phi = 6 \hat{i} + 4 \hat{j} + 2 \hat{k}$

If \hat{n} is a unit vector in outward normal to the sphere

then $\hat{n} = \frac{6 \hat{i} + 4 \hat{j} + 2 \hat{k}}{\sqrt{36 + 16 + 4}} = \frac{1}{\sqrt{56}} (6 \hat{i} + 4 \hat{j} + 2 \hat{k})$

\therefore Directional derivative of f in the outward normal to the

sphere = $\nabla f \cdot \hat{n}$

$$= (324 \hat{i} + 432 \hat{k}) \cdot \frac{1}{\sqrt{56}} (6 \hat{i} + 4 \hat{j} + 2 \hat{k}) = \frac{1944 + 864}{\sqrt{56}} = \frac{1404}{\sqrt{14}}$$

Q-2: Find the directional derivative of $\phi = (x^2 + y^2 + z^2)^{-1/2}$ at the point $(3, 1, 2)$ in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$.

Soln: $\phi = (x^2 + y^2 + z^2)^{-1/2}$

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-1/2} \quad | \quad (2013-14)$$

$$= \hat{i} \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x) \right\} + \hat{j} \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2y) \right\}$$

$$+ \hat{k} \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2z) \right\}$$

$$= - \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}} = - \frac{3\hat{i} + \hat{j} + 2\hat{k}}{14\sqrt{14}} \quad \text{at } (3, 1, 2)$$

Let \hat{a} be the unit vector in the given direction, then

$$\hat{a} = \frac{yz\hat{i} + zx\hat{j} + xy\hat{k}}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}} = \frac{2\hat{i} + 6\hat{j} + 3\hat{k}}{7} \quad | \quad \text{at } (3, 1, 2)$$

$$\begin{aligned}
 \therefore \text{Directional derivative} &= \hat{a} \cdot \nabla \phi \\
 &= \frac{2\hat{i} + 6\hat{j} + 3\hat{k}}{7} \cdot \left(-\frac{3\hat{i} + \hat{j} + 2\hat{k}}{14\sqrt{14}} \right) \\
 &= -\frac{6 + 6 + 6}{7 \cdot 14\sqrt{14}} = -\frac{9}{49\sqrt{14}}.
 \end{aligned}$$

Q.3: Find the directional derivative of $\left(\frac{1}{x^2}\right)$ in the direction of \vec{a} , where $\vec{a} = \hat{i}x + \hat{j}y + \hat{k}z$. (2016-17)

Solⁿ: $\nabla\left(\frac{1}{x^2}\right) = \frac{2}{x^3} \hat{i} = -\frac{2}{x^4} \cdot \vec{a}$ ($\because \hat{i} = \frac{\vec{a}}{|\vec{a}|}$)

Let \hat{e} be the unit vector in the direction of \vec{a} then $\hat{e} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{x}$

$$\therefore \text{Directional derivative} = \nabla\left(\frac{1}{x^2}\right) \cdot \hat{e} = -\frac{2}{x^4} \cdot \vec{a} \cdot \frac{\vec{a}}{x} = -\frac{2}{x^3}$$

Q.41: Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

(2018-19)

Solⁿ: $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$$\therefore \text{grad } \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(5x^2y - 5y^2z + \frac{5}{2}z^2x \right)$$

$$= \left(10xy + \frac{5}{2}z^2 \right) \hat{i} + (5x^2 - 10yz) \hat{j} + (-5y^2 + 5zx) \hat{k}$$

$$= \frac{25}{2} \hat{i} - 5 \hat{j} \quad \text{at } (1,1,1)$$

Here $\hat{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$

$$\therefore \text{Directional derivative} = (\text{grad } \phi) \cdot \hat{a}$$

$$= \left(\frac{25}{2} \hat{i} - 5 \hat{j} \right) \cdot \left(\frac{2}{3} \hat{i} - \frac{2}{3} \hat{j} + \frac{1}{3} \hat{k} \right)$$

$$= \frac{25}{3} + \frac{10}{3} = \frac{35}{3}$$

Q.5: Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find also the greatest rate of increase of ϕ . (2019-20)

Solⁿ: $\phi(x, y, z) = x^2yz + 4xz^2$

$$\therefore \nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2yz + 4xz^2)$$
$$= \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

At $(1, -2, 1)$, $\nabla\phi = \hat{j} + 6\hat{k}$

If \hat{n} is a unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$, then

$$\hat{n} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

So the required directional derivative at $(1, -2, 1)$

$$= \nabla\phi \cdot \hat{n} = (\hat{j} + 6\hat{k}) \cdot \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = |\hat{j} + 6\hat{k}| = \sqrt{1+36} = \sqrt{37}$

Practice Questions

1. Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$.

Ans: $-3\sqrt{2}$

2. Find the directional derivative of $\frac{1}{r}$ in the direction \bar{r} where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans: $\frac{1}{r^2}$

Practice Questions

3. Find the directional derivative of $f(x, y, z) = xyz$ at the point $P(1, -1, -2)$ in the direction of the vector $(2\hat{i} - 2\hat{j} + 2\hat{k})$.

Ans. $\frac{7}{3}$

4. Find the directional derivative of the scalar function of $f(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point $(3, 1, 3)$.

Ans. $\frac{27}{\sqrt{11}}$

Lecture 38

Divergence of a Vector Point Function

Definition

The divergence of a vector point function \vec{F} is denoted by $\text{div } \vec{F}$ and is defined as below.

Let
$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

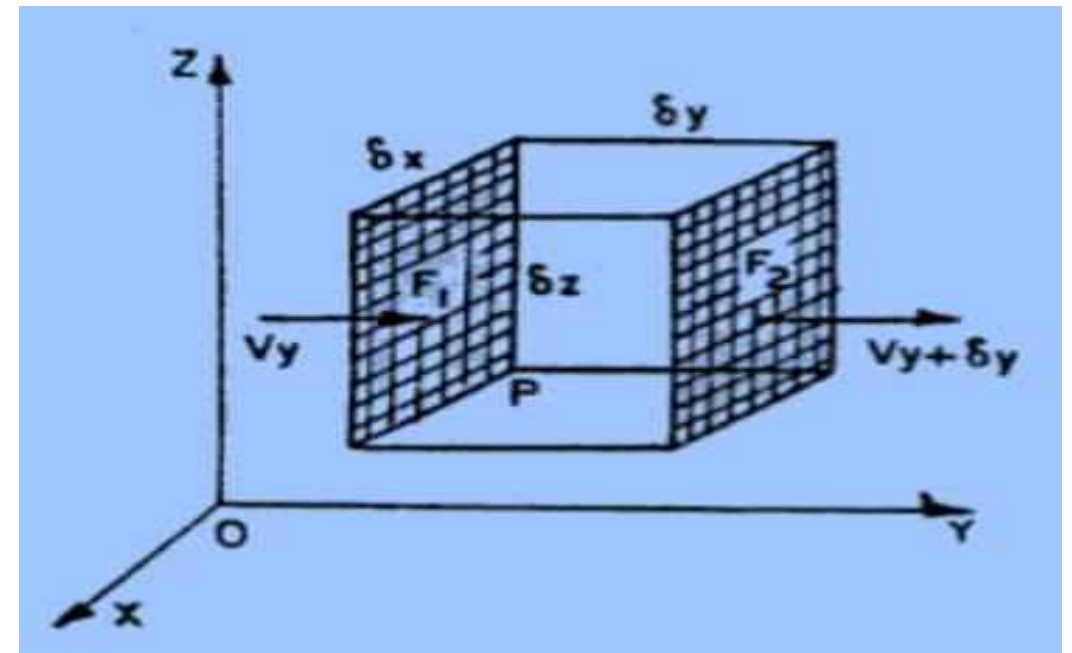
$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (\hat{i} F_1 + \hat{j} F_2 + \hat{k} F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\text{div } \vec{F}$ is a scalar function

Physical interpretation of Divergence of a Vector Function

Consider a fluid having density $\rho = \rho(x, y, z, t)$ and velocity $\vec{v} = \vec{v}(x, y, z, t)$ at a point (x, y, z) at time t . Let $\vec{V} = \rho\vec{v}$, then \vec{V} is a vector having the same direction as \vec{v} and magnitude $\rho |\vec{v}|$. It is known as *flux*. Its direction gives the direction of the fluid flow, and its magnitude gives the mass of the fluid crossing per unit time a unit area placed perpendicular to the direction of flow.

Consider the motion of the fluid having velocity $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ at a point $P(x, y, z)$. Consider a small parallelepiped with edges $\delta x, \delta y, \delta z$ parallel to the axes with one of its corners at P .

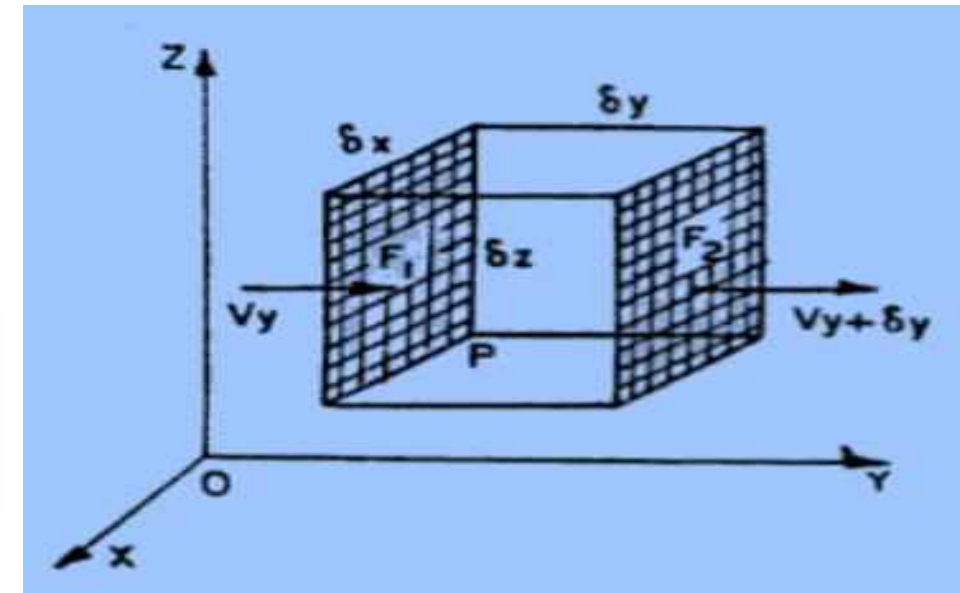


The mass of the fluid entering through the face F_1 per unit time is $V_y \delta x \delta z$ and that flowing out through the opposite face F_2 is $V_{y+\delta y} \delta x \delta z = \left(V_y + \frac{\partial V_y}{\partial y} \delta y \right) \delta x \delta z$ by using Taylor's series.

\therefore The net decrease in the mass of fluid flowing across these two faces

$$= \left(V_y + \frac{\partial V_y}{\partial y} \delta y \right) \delta x \delta z - V_y \delta x \delta z = \frac{\partial V_y}{\partial y} \delta x \delta y \delta z$$

Similarly, considering the other two pairs of faces, we get the total decrease in the mass of fluid inside the parallelepiped per unit time = $\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) \delta x \delta y \delta z$.



Dividing this by the volume $\delta x \delta y \delta z$ of the parallelepiped, we have the rate of loss of fluid per unit

volume

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \text{div } \vec{V}$$

Hence $\text{div } \vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

Thus it can be concluded that

$\text{div } \vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

If the fluid is incompressible, there can be no gain or loss in the volume element.

Hence $\text{div } \vec{V} = 0$ and \vec{V} is called a solenoidal vector function. Which is known in Hydrodynamics as the equation of continuity for incompressible fluids.

Note : Vectors having zero divergence are called solenoidal and are useful in various branches of physics and Engineering.

Example 1

Find the divergence of $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2, -1, 1)$.

Solution

$$\begin{aligned}\text{Div } \vec{V} &= \frac{\partial}{\partial x} (xyz) + \frac{\partial}{\partial y} (3x^2y) + \frac{\partial}{\partial z} (xz^2 - y^2z) \\ &= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)\end{aligned}$$

Example 2

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Prove that

$$(i) \operatorname{div} \vec{r} = 3 \text{ i.e., } \nabla \cdot \vec{r} = 3$$

$$(ii) \operatorname{div} (\vec{a} \times \vec{r}) = 0$$

Solution

$$(i) \operatorname{div} \vec{r} = \nabla \cdot \vec{r}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$\because \hat{i} \cdot \hat{i} = 1, \hat{i} \cdot \hat{j} = 0 \text{ etc.}$$

$$= 1 + 1 + 1$$

$$= 3$$

$$(ii) \Delta.(\vec{a} \times \vec{r}) = -\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) \cdot \{\hat{i} (a_3y - a_2z) - \hat{j} (a_3x - a_1z) + \hat{k} (a_2x - a_1y)\}$$

$$= -\frac{\partial}{\partial x} (a_3y - a_2z) + \frac{\partial}{\partial y} (a_3x - a_1z) - \frac{\partial}{\partial z} (a_2x - a_1y)$$

$$= 0$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{and } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

Example 3

If $\vec{v} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, find the value of $\text{div } \vec{v}$.

Solution

We have, $\vec{v} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\begin{aligned} \text{div } \vec{v} &= \vec{\nabla} \cdot \vec{v} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}} \right) \\ &= \frac{\partial}{\partial x} \frac{x}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial y} \frac{y}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{\left[(x^2 + y^2 + z^2)^{1/2} - x \cdot \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \cdot 2x \right]}{(x^2 + y^2 + z^2)} \end{aligned}$$

$$+ \frac{\left[(x^2 + y^2 + z^2)^{\frac{1}{2}} - y \cdot \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \times 2y \right]}{(x^2 + y^2 + z^2)} + \frac{\left[(x^2 + y^2 + z^2)^{1/2} - z \cdot \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \cdot 2z \right]}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2) - x^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(x^2 + y^2 + z^2) - y^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(x^2 + y^2 + z^2) - z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2}{\sqrt{(x^2 + y^2 + z^2)}}$$

Example 4

Prove that a vector field $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is solenoidal

Solution

A vector \vec{F} is said to be solenoidal if $\text{div } \vec{F} = 0$

Here $\text{div } \vec{F} = \nabla \cdot \vec{F}$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \{ (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j} \}$$

$$= \frac{\partial}{\partial x} (x^2 - y^2 + x) - \frac{\partial}{\partial y} (2xy + y)$$

$$= (2x + 1) - (2x + 1)$$

$$= 0$$

$$\Rightarrow \text{div } \vec{F} = 0$$

Thus the given vector is solenoidal

Example 5

If vector $\vec{F} = 3x \hat{i} + (x+y) \hat{j} - az \hat{k}$ is solenoidal. Find a.

Solution

A vector \vec{F} is said to be solenoidal, if $\text{div } \vec{F} = 0$

$$\therefore \text{div } \vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial x}(-az)$$

$$= 3 + 1 - a = 0$$

$$\therefore a = 4 \text{ Answer.}$$

Example 6

If r and \vec{r} have their usual meanings, show that $\operatorname{div} r^n \vec{r} = (n+3)r^n$

Solution

Since $\vec{r} = (x \hat{i} + y \hat{j} + z \hat{k})$ so; we have

$$r^n \vec{r} = r^n x \hat{i} + r^n y \hat{j} + r^n z \hat{k}$$

$$\therefore \operatorname{div} r^n \vec{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (r^n x \hat{i} + r^n y \hat{j} + r^n z \hat{k})$$

$$= \frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z)$$

$$= r^n \cdot 1 + nr^{n-1} \frac{\partial r}{\partial x} x + r^n \cdot 1 + nr^{n-1} \frac{\partial r}{\partial y} y + r^n \cdot 1 + nr^{n-1} z \frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1} \left(x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right)$$

$$= 3r^n + nr^{n-1} \left(x \frac{x}{r} + y \frac{y}{r} + z \frac{z}{r} \right)$$

$$= 3r^n + nr^{n-1} \left(\frac{x^2 + y^2 + z^2}{r} \right)$$

$$= 3r^n + nr^n$$

$$= (n+3)r^n$$

Example 7

Find $\text{div } \vec{F}$ where $F = \text{grad } (x^3 + y^3 + z^3 - 3xyz)$.

Ans. $\text{div } \vec{F} = 6(x + y + z)$, $\therefore \text{div } \vec{F} = 6(x + y + z)$

Q-6 Find the directional derivative of scalar function $f(x, y, z) = xyz$ at point $P(1, 1, 3)$ in the direction of the upward drawn normal to the sphere $x^2 + y^2 + z^2 = 11$ through the point P . (2022-23)

Soln $f = xyz$

$$\text{Now, } \nabla f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (xyz)$$

$$= yz \hat{i} + xz \hat{j} + xy \hat{k}$$

$$\text{at } P(1, 1, 3) \quad \nabla f = 3 \hat{i} + 3 \hat{j} + \hat{k}$$

Normal to the sphere $x^2 + y^2 + z^2 = 11$
 i.e., $\phi \equiv x^2 + y^2 + z^2 - 11$

$$\nabla \phi = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (x^2 + y^2 + z^2 - 11)$$

$$= (2x \hat{i} + 2y \hat{j} + 2z \hat{k})$$

At $P(1, 1, 3)$, $\nabla\phi = 2\hat{i} + 2\hat{j} + 6\hat{k}$
If \hat{n} is a unit vector normal to the sphere

$$\begin{aligned} \text{Then } \hat{n} &= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{4+4+36}} \\ &= \frac{2\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{44}} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}} \end{aligned}$$

\therefore Directional derivative of f in the upward normal to the sphere = $\nabla f \cdot \hat{n}$

$$= (3\hat{i} + 3\hat{j} + \hat{k}) \left(\frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}} \right)$$

$$= \frac{3+3+3}{\sqrt{11}} = \frac{9}{\sqrt{11}}$$

Practice Questions

1 Show that the vector $V = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$ is solenoidal.

2 Find $\text{div } \vec{F}$ where $F = \text{grad } (x^3 + y^3 + z^3 - 3xyz)$.

Ans. $\text{div } \vec{F} = 6(x + y + z)$.

3 If $u = x^2 + y^2 + z^2$, and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\text{div } (u\vec{r})$ in terms of u .

Ans : $5u$

4 If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (i) $\text{div } \left(\frac{\vec{r}}{r^3} \right) = 0$,

Q1 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (i) $\operatorname{div} \left(\frac{\vec{r}}{r^3} \right) = 0$,

Q2 If $u = x^2 + y^2 + z^2$, and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\operatorname{div} (u\vec{r})$ in terms of u .

Q3 Show that the vector $\vec{V} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$ is solenoidal.

Q4 A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$
Is motion possible for incompressible fluid?

Lecture 39

Curl of a Vector Point Function & Vector Identities

Curl of a Vector Point Function

The curl of a vector point function is a vector quantity if $\bar{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$
Then

The curl (or rotation) of \bar{V} is denoted by $\text{curl } \bar{V}$ and is defined as

$$\text{curl } \bar{V} = \nabla \times \bar{V} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \hat{j} \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \hat{k} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

Physical Interpretation

Consider a rigid body rotating about a given axis through O with uniform angular velocity ω .

$$\text{Let } \vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

The linear velocity \vec{V} of any point P(x, y, z) on the rigid body is given by

$$\vec{V} = \vec{\omega} \times \vec{r}$$

Where $\vec{r} = \hat{i} x + \hat{j} y + \hat{k} z$ is the position vector of P

$$\therefore \vec{V} = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

$$\therefore \text{curl } \bar{V} = \text{curl} (\bar{\omega} \times \bar{r}) = \nabla \times (\bar{\omega} \times \bar{r})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$$

$\therefore \omega_1, \omega_2, \omega_3$ are constants

$$= 2 \bar{\omega}$$

$$\therefore \bar{\omega} = \frac{1}{2} \text{curl } \bar{V}$$

Thus the angular velocity at any points is equal to half the curl of linear velocity at that point of the body.

Note : If $\text{curl } \bar{V} = 0$, then \bar{V} is said to be an irrotational vector, otherwise rotational. Also curl of a vector signifies rotation.

Example 1

Find the curl of $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at $(2, -1, 1)$

Solution

Here, we have

$$\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = -2yz\hat{i} - (z^2 - xy)\hat{j} + (6xy - xz)\hat{k}$$

$$= -2yz\hat{i} + (xy - z^2)\hat{j} + (6xy - xz)\hat{k}$$

Curl at $(2, -1, 1)$

$$= -2(-1)(1)\hat{i} + \{(2)(-1) - 1\}\hat{j} + \{6(2)(-1) - 2(1)\}\hat{k} = 2\hat{i} - 3\hat{j} - 14\hat{k}$$

Example 2

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Prove that

(i) $\text{Curl } \vec{r} = \vec{0}$ i.e, $\nabla \times \vec{r} = \vec{0}$

(ii) $\text{Curl } (\vec{r} \times \vec{a}) = -2\vec{a}$ i.e, $\nabla \times (\vec{r} \times \vec{a}) = -2\vec{a}$

Solution

(i) $\text{Curl } \vec{r} = \nabla \times \vec{r}$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (x\hat{i} + y\hat{j} + z\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \hat{j} \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \hat{k} \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = \vec{0}$$

(ii) Let us suppose that

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{and } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\therefore \vec{r} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \hat{i} (a_3y - a_2z) - \hat{j} (a_3x - a_1z) + \hat{k} (a_2x - a_1y)$$

Therefore, we have

$$\nabla \times (\vec{r} \times \vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$$= \hat{i} (-a_1 - a_1) - \hat{j} (a_2 + a_2) + \hat{k} (-a_3 - a_3)$$

$$= -2a_1\hat{i} - 2a_2\hat{j} - 2a_3\hat{k}$$

$$= -2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$= -2\vec{a}$$

Example 3

A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$, show that the motion is irrotational and hence find velocity potential.

Solution

We have $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = (1-1)\hat{i} + (1-1)\hat{j} + (1-1)\hat{k} = \vec{0}$$

Hence \vec{V} is irrotational

Now, if ϕ is a scalar potential then, we have

$$\vec{V} = \nabla\phi$$

$$\Rightarrow (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of \hat{i} , \hat{j} , \hat{k} we get

$$\frac{\partial \phi}{\partial x} = y+z, \quad \frac{\partial \phi}{\partial y} = z+x \quad \& \quad \frac{\partial \phi}{\partial z} = x+y$$

$$\text{Also } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= (y+z)dx + (z+x)dy + (x+y)dz$$

$$= ydx + zdx + zdy + xdy + xdz + ydz$$

$$= ydx + xdy + zdy + ydz + xdz + zdx$$

$$= d(xy) + d(yz) + d(xz)$$

Integrating term by term we get

$$\phi = xy + yz + xz + \text{constant}$$

Example 4

Find the constants a , b , c , so that

$$\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

is irrotational.

Solution

We have,

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + 2y + az) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} \\ &= (c + 1)\hat{i} - (4 - a)\hat{j} + (b - 2)\hat{k} \end{aligned}$$

As \vec{F} is irrotational, $\nabla \times \vec{F} = \vec{0}$

$$\text{i.e., } (c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\therefore c+1 = 0, \quad 4-a = 0 \quad \text{and} \quad b-2 = 0$$

$$\text{i.e., } a = 4, \quad b = 2, \quad c = -1$$

Putting the values of a, b, c in (1), we get

$$\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

Vector Identities

1. $\text{div} (\mathbf{A} + \mathbf{B}) = \text{div} \mathbf{A} + \text{div} \mathbf{B}$
2. $\text{curl} (\mathbf{A} + \mathbf{B}) = \text{curl} \mathbf{A} + \text{curl} \mathbf{B}$
3. *If \mathbf{A} is a differentiable vector function and ϕ is a differentiable scalar function, then*
$$\text{div} (\phi \mathbf{A}) = (\text{grad } \phi) \cdot \mathbf{A} + \phi \text{div} \mathbf{A}$$
4. $\text{curl} (\phi \mathbf{A}) = (\text{grad } \phi) \times \mathbf{A} + \phi \text{curl} \mathbf{A}$
5. $\text{div} (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl} \mathbf{A} - \mathbf{A} \cdot \text{curl} \mathbf{B}$

Vector Identities

$$6. \quad \nabla \times (A \times B) = (B \cdot \nabla)A - B(\nabla \cdot A) - (A \cdot \nabla)B + A(\nabla \cdot B)$$

$$7. \quad \nabla \times (\nabla f) = 0$$

$$8. \quad \nabla \cdot (\nabla \times A) = 0$$

$$9. \quad \nabla^2 f \triangleq \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Example 5

(i) Prove that the vector $f(r)\vec{r}$ is irrotational

(ii) Prove that the vector $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$ is irrotational

Solution

$$\begin{aligned}
 (i) \text{ curl } \{f(r)\vec{r}\} &= f(r) \text{ curl } \vec{r} + \{\text{grad } f(r)\} \times \vec{r} \\
 &= \vec{0} + f'(r) \hat{r} \times \vec{r} \\
 &= \frac{f'(r)}{r} (\vec{r} \times \vec{r}) = \vec{0}
 \end{aligned}$$

Hence $f(r)\vec{r}$ is irrotational.

$$(ii) \quad \text{grad } f(r) = f'(r)\hat{r} = \frac{1}{r} f'(r) \vec{r}$$

$$\text{div } \{\text{grad } f(r)\} = \nabla^2 f(r)$$

$$= \text{div} \left\{ \frac{f'(r)}{r} \vec{r} \right\} = \frac{f'(r)}{r} \text{div } \vec{r} + \text{grad} \left\{ \frac{f'(r)}{r} \right\} \cdot \vec{r}$$

$$= \frac{3}{r} f'(r) + \left\{ \frac{rf''(r) - f'(r)}{r^2} \right\} \hat{r} \cdot \vec{r}$$

$$= \frac{3}{r} f'(r) + \left\{ \frac{rf''(r) - f'(r)}{r^3} \right\} (\vec{r} \cdot \vec{r})$$

$$= \frac{3}{r} f'(r) + \left\{ \frac{rf''(r) - f'(r)}{r} \right\}$$

$$\Rightarrow \quad \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

$$\text{Now,} \quad \nabla^2 \log r = -\frac{1}{r^2} + \frac{2}{r} \left(\frac{1}{r} \right) = \frac{1}{r^2} = \frac{1}{x^2 + y^2 + z^2}$$

Q. \rightarrow If $\vec{A} = (xz^2\hat{i} + 2y\hat{j} - 3xz\hat{k})$ and $\vec{B} = (3xz\hat{i} + 2yz\hat{j} - z^2\hat{k})$.

Find the value of $[\vec{A} \times (\nabla \times \vec{B})]$ & $[(\vec{A} \times \nabla) \times \vec{B}]$. (2016-17)

Solⁿ: (i) $[\vec{A} \times (\nabla \times \vec{B})]$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2yz & -z^2 \end{vmatrix} = \hat{i}[0 - 2y] - \hat{j}[0 - 3x] + \hat{k}[0 - 0]$$
$$= -2y\hat{i} + 3x\hat{j} + 0\hat{k}$$

Now

$$[\vec{A} \times (\nabla \times \vec{B})] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ -2y & 3x & 0 \end{vmatrix} = \hat{i}[0 + 9xz^2] - \hat{j}[0 - 6xy^2] + \hat{k}[3xz^2 + 4y^2]$$
$$= 9xz^2\hat{i} + 6xy^2\hat{j} + (3xz^2 + 4y^2)\hat{k}$$

$$(ii) [(\vec{A} \times \nabla) \times \vec{B}]$$

$$\vec{A} \times \nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}[0+0] - \hat{j}[2xz+3z] + \hat{k}[0-0]$$
$$= 0\hat{i} - (2xz+3z)\hat{j} + 0\hat{k}$$

$$[(\vec{A} \times \nabla) \times \vec{B}] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -(2xz+3z) & 0 \\ 3xz & 2yz & -z^2 \end{vmatrix} = [z^2(2xz+3z)-0]\hat{i} - [0-0]\hat{j}$$
$$+ [0+3xz(2xz+3z)]\hat{k}$$
$$= (2xz^3+3z^3)\hat{i} + (6x^2z^2+9xz^2)\hat{k}$$

Q. \Rightarrow A fluid motion is given by

$$\vec{v} = (y \sin z - \sin x) \hat{i} + (x \sin z + 2yz) \hat{j} + (xy \cos z + y^2) \hat{k}.$$

Is the motion irrotational? If so, find the velocity potential.

(2020-21)

Solⁿ: Proceed as



Q. \Rightarrow If $\vec{F} = (\vec{e} \cdot \vec{r}) \vec{r}$, where \vec{e} is a constant vector, find curl \vec{F} and prove that it is perpendicular to \vec{e} . (2011-12)

Solⁿ: Let $\vec{e} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$

Now

$$\vec{e} \cdot \vec{r} = a_1 x + a_2 y + a_3 z$$

$$\begin{aligned} \Rightarrow (\vec{e} \cdot \vec{r}) \vec{r} &= (a_1 x + a_2 y + a_3 z) \cdot (x \hat{i} + y \hat{j} + z \hat{k}) \\ &= (a_1 x^2 + a_2 xy + a_3 xz) \hat{i} + (a_1 xy + a_2 y^2 + a_3 yz) \hat{j} \\ &\quad + (a_1 xz + a_2 yz + a_3 z^2) \hat{k} \end{aligned}$$

Now $\text{Curl}(\vec{a} \cdot \vec{a}) \cdot \vec{a} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1x^2 + a_2xy + a_3xz & a_1xy + a_2y^2 + a_3yz & a_1xz + a_2yz + a_3z^2 \end{vmatrix}$$

$$\text{Curl } \vec{F} = \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

Now to show $\text{curl } \vec{F}$ is perpendicular to \vec{a} i.e. we have to show $\text{curl } \vec{F} \cdot \vec{a} = 0$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\begin{aligned} \text{curl } \vec{F} \cdot \vec{a} &= (a_2z - a_3y)a_1 - (a_1z - a_3x)a_2 + (a_1y - a_2x)a_3 \\ &= 0 \end{aligned}$$

Practice Questions

- (1) Find the Curl of the following vector fields

$$\vec{F} = x^2y^2\hat{i} + 2xy\hat{j} - (y^2 - xy)\hat{k} \quad \text{at } (1,2,3)$$

$$\text{Ans: Curl } \vec{F} = (2y - x)\hat{i} + y\hat{j} + 2y(1 - x^2)\hat{k}$$

- (2) Find the Curl of the following vector fields

$$\vec{F} = e^{xyz}(xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k})$$

$$\text{Ans: Curl } \vec{F} = -39e^6\hat{i} + 3e^6\hat{j} + 92e^6\hat{k}$$

- (3) If a vector field is given by

$$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}. \text{ Is this field irrotational? If so, find its scalar potential.}$$

$$\text{Ans: Yes, scalar potential is } \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$$

Practice Questions

(4)

A fluid motion is given by

$$\vec{v} = (y \sin z - \sin x) \hat{i} + (x \sin z + 2yz) \hat{j} + (xy \cos z + y^2) \hat{k}$$

is the motion irrotational? If so, find the velocity potential.

Ans: Yes, **Velocity potential** = $xy \sin z + \cos x + y^2 z + c.$

Lecture 40

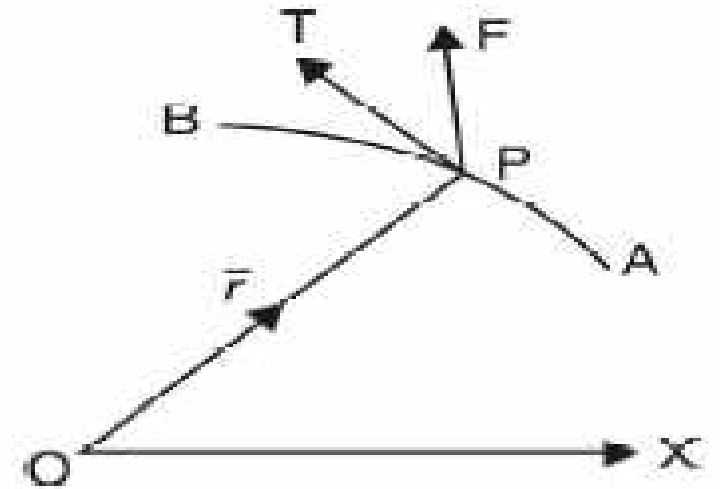
Line Integral, Surface Integral And Volume Integral

Line Integral

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB .

Line integral of a vector function \vec{F} along the curve AB is defined as

$$\text{Line integral} = \int_c \left(\vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int_c \vec{F} \cdot d\vec{r}$$



Note (1) Work. If \vec{F} represents the variable force acting on a particle along arc AB , then the

$$\text{total work done} = \int_A^B \vec{F} \cdot d\vec{r}$$

Example 1

If a force $\vec{F} = 2x^2 y \hat{i} + 3xy \hat{j}$ displaces a particle in the xy -plane from $(0, 0)$ to $(1, 4)$ along a curve $y = 4x^2$. Find the work done.

Solution

$$\begin{aligned}
 \text{Work done} &= \int_c \vec{F} \cdot d\vec{r} && \left[\begin{array}{l} \vec{r} = x\hat{i} + y\hat{j} \\ d\vec{r} = dx\hat{i} + dy\hat{j} \end{array} \right] \\
 &= \int_c (2x^2 y \hat{i} + 3xy \hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\
 &= \int_c (2x^2 y dx + 3xy dy)
 \end{aligned}$$

Putting the values of y and dy , we get

$$\begin{pmatrix} y = 4x^2 \\ dy = 8x dx \end{pmatrix}$$

$$= \int_0^1 \cdot [2x^2 (4x^2) dx + 3x (4x^2) 8x dx]$$

$$= 104 \int_0^1 x^4 dx = 104 \left(\frac{x^5}{5} \right)_0^1 = \frac{104}{5}$$

Ans.

Example 2

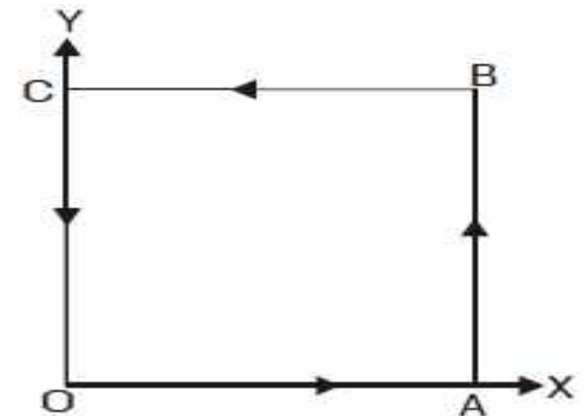
Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$ and $y = a$.

Solution

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

$$\vec{r} = x\hat{i} + y\hat{j}, \quad d\vec{r} = dx\hat{i} + dy\hat{j}, \quad \vec{F} = x^2\hat{i} + xy\hat{j}$$

$$\vec{F} \cdot d\vec{r} = x^2 dx + xy dy \quad \dots(1)$$



On OA , $y = 0$

$$\therefore \vec{F} \cdot \vec{dr} = x^2 dx$$

$$\int_{OA} \vec{F} \cdot \vec{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3} \right]_0^a = \frac{a^3}{3} \quad \dots(2)$$

On AB , $x = a$
(1) becomes

$$\therefore dx = 0$$

$$\therefore \vec{F} \cdot \vec{dr} = ay dy$$

$$\int_{Ab} \vec{F} \cdot \vec{dr} = \int_0^a ay dy = a \left[\frac{y^2}{2} \right]_0^a = \frac{a^3}{2}$$

... (3)

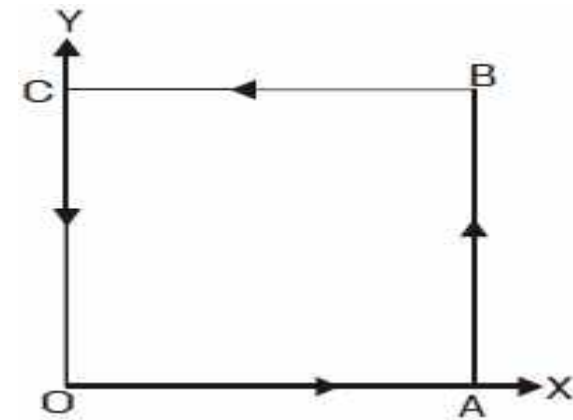
On BC , $y = a$

$$\therefore dy = 0$$

\Rightarrow (1) becomes

$$\vec{F} \cdot \vec{dr} = x^2 dx$$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_a^0 x^2 dx = \left[\frac{x^3}{3} \right]_a^0 = \frac{-a^3}{3} \quad \dots(4)$$



On CO , $x = 0$, $\therefore \vec{F} \cdot \vec{dr} = 0$
(1) becomes

$$\int_{CO} \vec{F} \cdot \vec{dr} = 0 \quad \dots(5)$$

On adding (2), (3), (4) and (5), we get $\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$

Q.1: Find the work done in moving a particle in the force field:

$$\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k} \text{ along the curve } x^2 = 4y \text{ and } 3x^3 = 8z$$

from $x=0$ to $x=2$.

(2011-12)

Solⁿ: Work done = $\int_C \vec{F} \cdot d\vec{r}$

$$= \int_C [3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C 3x^2 dx + (2xz - y) dy + z dz$$

Along the curve $x^2 = 4y$ and $3x^3 = 8z$ from $x=0$ to $x=2$

$$\Rightarrow 2x dx = 4 dy \text{ and } 9x^2 dx = 8 dz$$

$$\Rightarrow \text{Work done} = \int_{x=0}^2 \left[3x^2 dx + \left(2x \cdot \frac{3x^3}{8} - \frac{x^2}{4} \right) \frac{x}{2} dx + \frac{3x^3}{8} \cdot \frac{9x^2}{8} dx \right]$$

$$= \int_{x=0}^2 \left[3x^2 + \frac{1}{8} (3x^5 - x^3) + \frac{27}{64} x^5 \right] dx$$

$$= \left[x^3 + \frac{1}{8} \left(\frac{3x^6}{6} - \frac{x^4}{4} \right) + \frac{27}{64} \frac{x^6}{6} \right]_0^2$$

$$= \left[8 + \frac{1}{8} (32 - 4) + \frac{27}{64} \cdot \frac{64}{6} \right] = 16$$

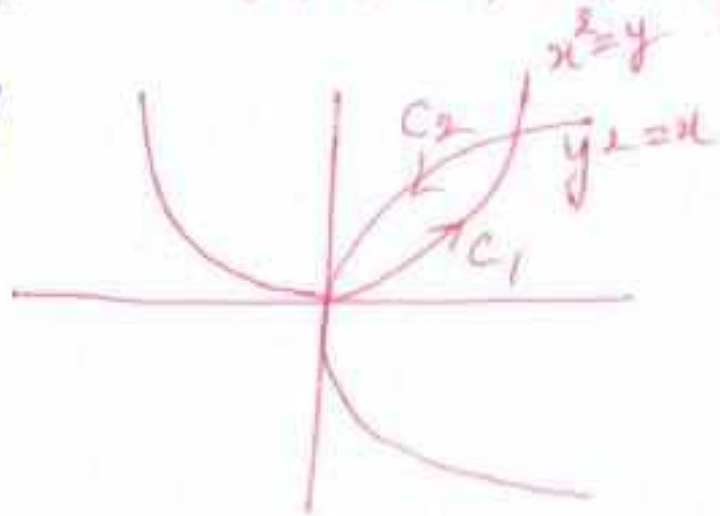
Q.31 \rightarrow If $\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$, evaluate $\oint_C \vec{A} \cdot d\vec{s}$ around the curve C consisting of $y=x^2$ and $y^2=x$.

(2013-14, 2017-18)

Solⁿ \rightarrow $\oint_C \vec{A} \cdot d\vec{s} = \oint_C (x-y) dx + (x+y) dy$

C consisting of $y=x^2$ and $y^2=x$

$$\oint_C \vec{A} \cdot d\vec{s} = \oint_{C_1} \vec{A} \cdot d\vec{s} + \oint_{C_2} \vec{A} \cdot d\vec{s}$$



Along C_1 , $y=x^2$, $dy=2x dx$

$$\Rightarrow \oint_{C_1} \vec{A} \cdot d\vec{s} = \int_0^1 (x-x^2) dx + (x+x^2) 2x dx = \int_0^1 (x+x^2+2x^3) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^4}{4} \right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{4}{3}$$

Now Along C_2 , $x=y^2$, $dx=2y dy$

$$\oint_{C_2} \vec{A} \cdot d\vec{x} = \int_1^0 (y^2 - y) 2y dy + (y^2 + y) dy$$

$$\oint_{C_2} \vec{A} \cdot d\vec{x} = \int_1^0 (2y^3 - y^2 + y) dy = \left[\frac{y^4}{2} - \frac{y^3}{3} + \frac{y^2}{2} \right]_1^0$$

$$= - \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{2} \right] = - \frac{2}{3}$$

Thus $\oint_C \vec{A} \cdot d\vec{x} = \frac{4}{3} + \left(-\frac{2}{3} \right) = \frac{2}{3}$

Surface Integral

Surface integral of a vector function \vec{F} over the surface S is defined as the integral of the components of \vec{F} along the normal to the surface.

Component of \vec{F} along the normal

= $\vec{F} \cdot \hat{n}$, where \hat{n} is the unit normal vector to an element ds and

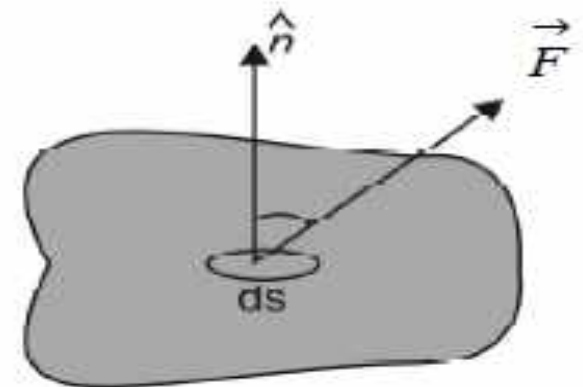
$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|}$$

$$ds = \frac{dx \, dy}{(\hat{n} \cdot \hat{k})}$$

Surface integral of F over S

$$= \int \vec{F} \cdot \hat{n}$$

$$= \iint_S (\vec{F} \cdot \hat{n}) \, ds$$



Example 3

• Evaluate $\iint_S \vec{A} \cdot \hat{n} \, ds$ where $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Solution

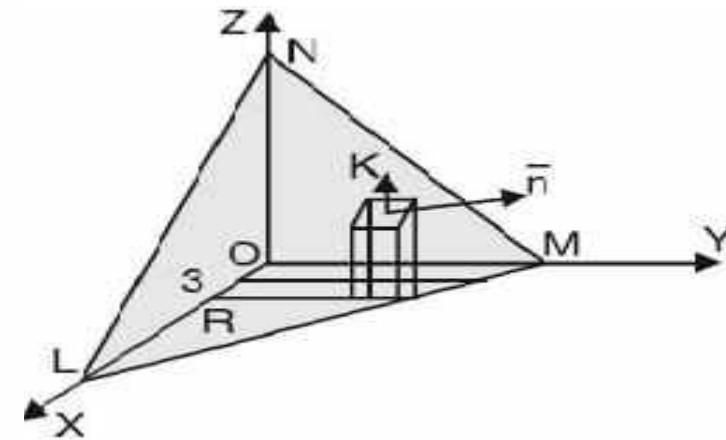
A vector normal to the surface "S" is given by

$$\nabla(2x + y + 2z) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

And \hat{n} = a unit vector normal to surface S

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

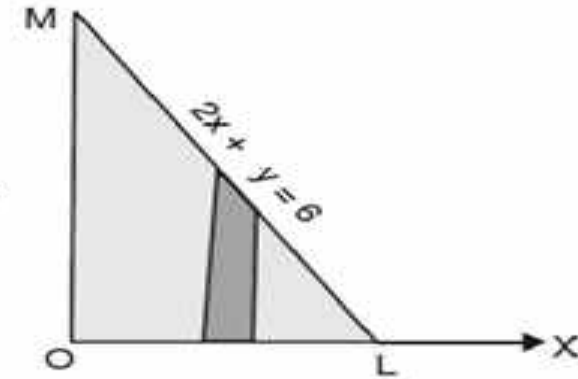
$$\hat{k} \cdot \hat{n} = \hat{k} \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k} \right) = \frac{2}{3}$$



$$\therefore \iint_S \vec{A} \cdot \hat{n} \, ds = \iint_R \vec{A} \cdot \hat{n} \frac{dx \, dy}{\hat{k} \cdot \vec{n}}$$

Where R is the projection of S .

$$\begin{aligned} \text{Now, } \vec{A} \cdot \hat{n} &= [(x + y^2) \hat{i} - 2x\hat{j} + 2yz\hat{k}] \cdot \left(\frac{2}{3} \hat{i} + \frac{1}{3} \hat{j} + \frac{2}{3} \hat{k} \right) \\ &= \frac{2}{3} (x + y^2) - \frac{2}{3} x + \frac{4}{3} yz = \frac{2}{3} y^2 + \frac{4}{3} yz \end{aligned}$$



... (1)

Putting the value of z in (1), we get

$$\vec{A} \cdot \hat{n} = \frac{2}{3} y^2 + \frac{4}{3} y \left(\frac{6 - 2x - y}{2} \right) \left(\begin{array}{l} \because \text{on the plane } 2x + y + 2z = 6, \\ z = \frac{(6 - 2x - y)}{2} \end{array} \right)$$

$$\vec{A} \cdot \hat{n} = \frac{2}{3} y (y + 6 - 2x - y) = \frac{4}{3} y (3 - x) \quad \dots (2)$$

Hence,
$$\iint_S \vec{A} \cdot \hat{n} ds = \iint_R \bar{A} \cdot \bar{n} \frac{dx dy}{|\hat{k} \cdot \bar{n}|} \quad \dots(3)$$

Putting the value of $\vec{A} \cdot \hat{n}$ from (2) in (3), we get

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} ds &= \iint_R \frac{4}{3} y (3-x) \cdot \frac{3}{2} dx dy = \int_0^3 \int_0^{6-2x} 2y (3-x) dy dx \\ &= \int_0^3 2 (3-x) \left[\frac{y^2}{2} \right]_0^{6-2x} dx \\ &= \int_0^3 (3-x) (6-2x)^2 dx = 4 \int_0^3 (3-x)^3 dx \\ &= 4 \cdot \left[\frac{(3-x)^4}{4(-1)} \right]_0^3 = -(0-81) = 81 \end{aligned}$$

Example 4

Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot d\vec{s}$ where S is the surface of the sphere

$x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution

Here, $\phi = x^2 + y^2 + z^2 - a^2$

$$\begin{aligned}
 \text{Vector normal to the surface} &= \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \\
 &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \\
 \hat{n} &= \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \\
 &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \quad [\because x^2 + y^2 + z^2 = a^2]
 \end{aligned}$$

Here,

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{F} \cdot \hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \right) = \frac{3xyz}{a}$$

$$\begin{aligned} \text{Now, } \iint_S \vec{F} \cdot \hat{n} \, ds &= \iint_S (\vec{F} \cdot \hat{n}) \frac{dx \, dy}{|\hat{k} \cdot \hat{n}|} = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{3xyz \, dx \, dy}{a \left(\frac{z}{a} \right)} \\ &= 3 \int_0^a \int_0^{\sqrt{a^2 - x^2}} xy \, dy \, dx = 3 \int_0^a x \left(\frac{y^2}{2} \right)_0^{\sqrt{a^2 - x^2}} dx \\ &= \frac{3}{2} \int_0^a x (a^2 - x^2) \, dx = \frac{3}{2} \left(\frac{a^2 x^2}{2} - \frac{x^4}{4} \right)_0^a = \frac{3}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{3a^4}{8}. \end{aligned}$$

Volume Integral

Let \vec{F} be a vector point function and volume V enclosed by a closed surface.

The volume integral = $\iiint_V \vec{F} dV$

Example 5

If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F} \, dv$ where, v is the region bounded by the surfaces
 $x = 0, y = 0, x = 2, y = 4, z = x^2, z = 2.$

Solution

$$\begin{aligned}
 \iiint_V \vec{F} \, dv &= \iiint (2z\hat{i} - x\hat{j} + y\hat{k}) \, dx \, dy \, dz \\
 &= \int_0^2 \int_0^4 \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k}) \, dz \, dy \, dx = \int_0^2 \int_0^4 [z^2\hat{i} - xz\hat{j} + yz\hat{k}]_{x^2}^2 \, dy \, dx \\
 &= \int_0^2 \int_0^4 [4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}] \, dy \, dx \\
 &= \int_0^2 \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k} \right]_0^4 \, dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}) dx \\
 &= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k} \right]_0^2 \\
 &= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})
 \end{aligned}$$

Practice Questions

- 1 Find the work done by a force $y\hat{i} + x\hat{j}$ which displaces a particle from origin to a point $(\hat{i} + \hat{j})$
Ans. 1
- 2 Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from origin to $(1, 1)$ along a parabola $y^2 = x$.
Ans. $\frac{2}{3}$
- 3 If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \times \vec{F} dV$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.
Ans. $\frac{8}{3}(\hat{j} - \hat{k})$
- 4 Evaluate $\iint_S \vec{A} \cdot \hat{n} ds$, where $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.
Ans. 81

Practice Questions

- 5 Evaluate $\iint_S \vec{A} \cdot \hat{n} \, ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. Ans. 90
- 6 Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$, where, $F = 2yx\hat{i} - yz\hat{j} + x^2\hat{k}$ over the surface S of the cube bounded by the coordinate planes and planes $x = a$, $y = a$ and $z = a$. Ans. $\frac{1}{2}a^4$

Thank You

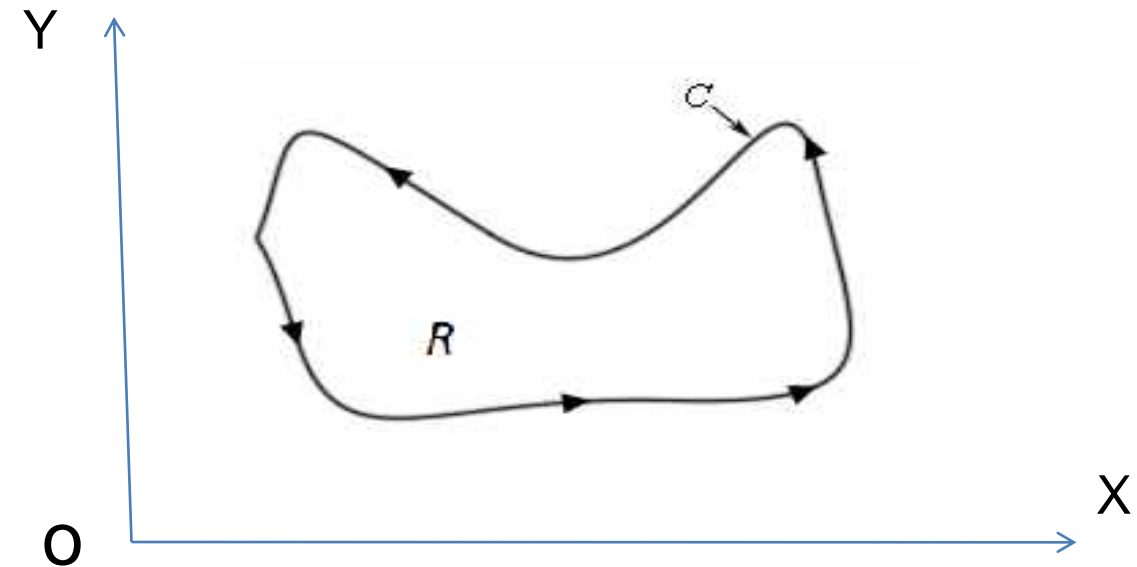
Lecture 41 (I)

Green's Theorem and its Applications - I

Green Theorem

If $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in $x - y$ plane, then

$$\oint_C (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy$$



Example 1

Using Green's Theorem, evaluate $\int_c (x^2 y dx + x^2 dy)$, where c is the boundary described counter clockwise of the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

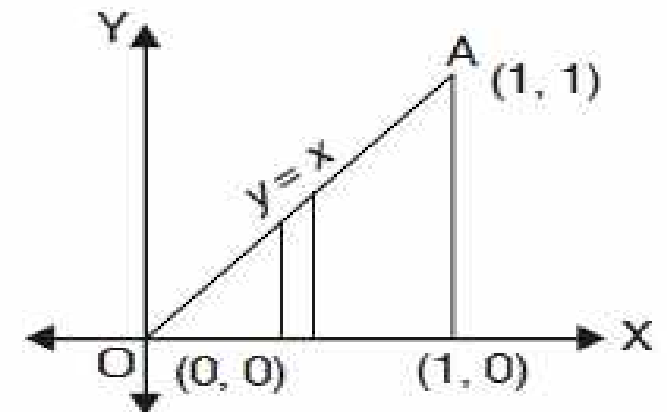
Solution

By Green's Theorem, we have

$$\int_c (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy$$

$$\int_c (x^2 y dx + x^2 dy) = \iint_R (2x - x^2) dx dy$$

$$= \int_0^1 (2x - x^2) dx \int_0^x dy = \int_0^1 (2x - x^2) dx [y]_0^x$$



$$\begin{aligned} &= \int_0^1 (2x - x^2)(x) dx = \int_0^1 (2x^2 - x^3) dx = \left(\frac{2x^3}{3} - \frac{x^4}{4} \right)_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5}{12} \end{aligned}$$

Example 2

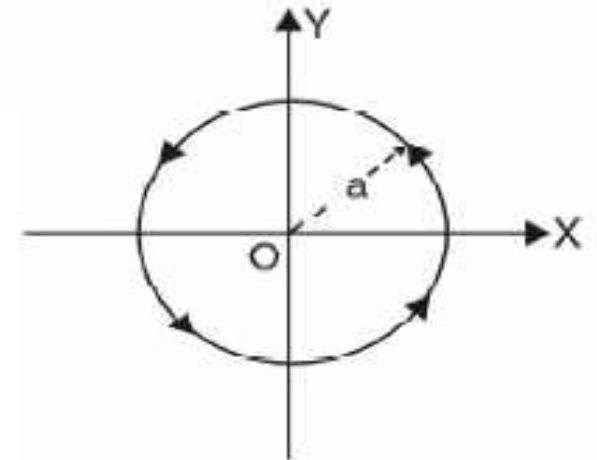
A vector field \vec{F} is given by $\vec{F} = \sin y \hat{i} + x(1 + \cos y) \hat{j}$.

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the circular path given by $x^2 + y^2 = a^2$.

Solution

$$\vec{F} = \sin y \hat{i} + x(1 + \cos y) \hat{j}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [\sin y \hat{i} + x(1 + \cos y) \hat{j}] \cdot (\hat{i}dx + \hat{j}dy) = \int_C \sin y dx + x(1 + \cos y) dy$$



On applying Green's Theorem, we have

$$\begin{aligned}\oint_c (\phi dx + \psi dy) &= \iint_s \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy \\ &= \iint_s [(1 + \cos y) - \cos y] dx dy\end{aligned}$$

where s is the circular plane surface of radius a .

$$= \iint_s dx dy = \text{Area of circle} = \pi a^2.$$

Example 3

Apply Green's Theorem to evaluate $\int_C [(2x^2 - y^2) dx + (x^2 + y^2) dy]$, where C is the boundary of the area enclosed by the x -axis and the upper half of circle $x^2 + y^2 = a^2$.

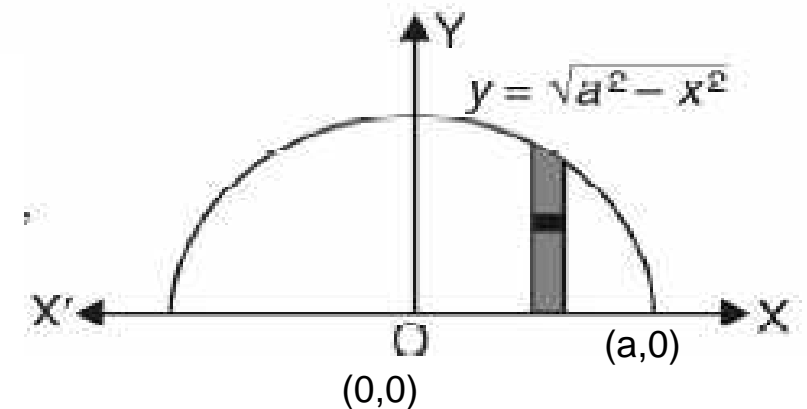
Solution

$$\int_C [(2x^2 - y^2) dx + (x^2 + y^2) dy]$$

By Green's Theorem, we've $\int_C (\phi dx + \psi dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy$

$$= \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} \left[\frac{\partial}{\partial x} (x^2 + y^2) - \frac{\partial}{\partial y} (2x^2 - y^2) \right] dx dy$$

$$= \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} (2x + 2y) dx dy = 2 \int_{-a}^a dx \int_0^{\sqrt{a^2 - x^2}} (x + y) dy$$



$$= 2 \int_{-a}^a dx \left(xy + \frac{y^2}{2} \right) \Big|_0^{\sqrt{a^2 - x^2}} = 2 \int_{-a}^a \left(x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2} \right) dx$$

$$= 2 \int_{-a}^a x\sqrt{a^2 - x^2} dx + \int_{-a}^a (a^2 - x^2) dx \quad \left[\begin{array}{l} \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, f \text{ is even} \\ = 0, f \text{ is odd} \end{array} \right]$$

$$= 0 + 2 \int_0^a (a^2 - x^2) dx = 2 \left(a^2 x - \frac{x^3}{3} \right) \Big|_0^a = 2 \left(a^3 - \frac{a^3}{3} \right) = \frac{4a^3}{3}$$

Along $C_3 : y = x, dy = dx ; x : 1 \text{ to } 0 ;$

$$I_3 = \int_{C_3} (x dy - y dx) = \int (x dx - x dx) = 0$$

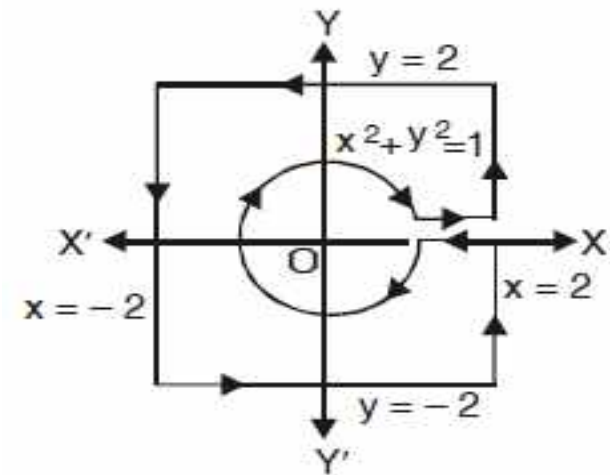
$$A = \frac{1}{2} (I_1 + I_2 + I_3) = \frac{1}{2} (0 + 2 \log 2 + 0) = \log 2$$

Example 4

Evaluate $\oint_C -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$, where $C = C_1 \cup C_2$ with $C_1 : x^2 + y^2 = 1$ and $C_2 : x = \pm 2, y = \pm 2$.

Solution

$$\begin{aligned} & \oint_C -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \iint \left(\frac{\partial}{\partial x} \frac{x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{y}{x^2 + y^2} \right) dx dy \\ &= \iint \left[\frac{(x^2 + y^2)1 - 2x(x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)1 - 2y(y)}{(x^2 + y^2)^2} \right] dx dy \end{aligned}$$



$$= \iint \left[\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \right] dx dy$$

$$= \iint \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} \right] dx dy$$

$$= \iint \frac{0}{(x^2 + y^2)^2} dx dy = 0$$

Example 1

Evaluate by Green's theorem $\int_C [e^{-x} \sin y dx + e^{-x} \cos y dy]$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi, \pi/2)$, $(0, \pi/2)$ and hence verify Green's theorem.

Solution

By Green's theorem we have

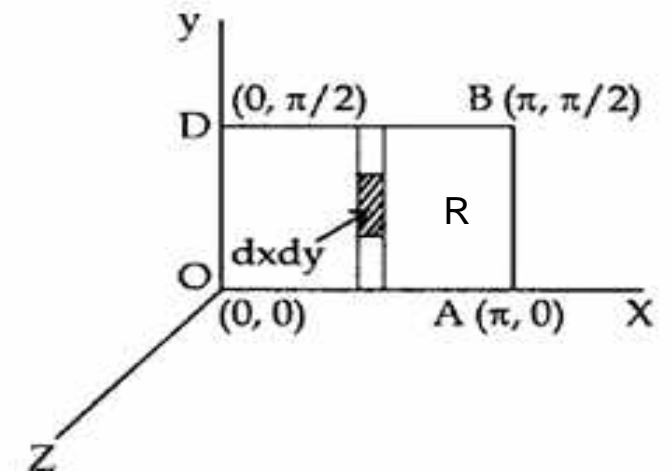
$$\int_C (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Comparing the given integral

$$M = e^{-x} \sin y \quad \text{and} \quad N = e^{-x} \cos y$$

$$\text{Then } \frac{\partial M}{\partial y} = e^{-x} \cos y \quad \text{and} \quad \frac{\partial N}{\partial x} = -e^{-x} \cos y$$

Hence by Green's theorem



$$\begin{aligned}
 \int_C [e^{-x} \sin y dx + e^{-x} \cos y dy] &= \iint_R (-e^{-x} \cos y - e^{-x} \cos y) dx dy \\
 &= -2 \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} e^{-x} \cos y dx dy \\
 &= -2 \left[-e^{-x} \right] \left[\sin y \right]_0^{\pi/2} \\
 &= 2(e^{-\pi} - 1)(1) \\
 &= 2(e^{-\pi} - 1)
 \end{aligned}$$

Evaluation of line integral:

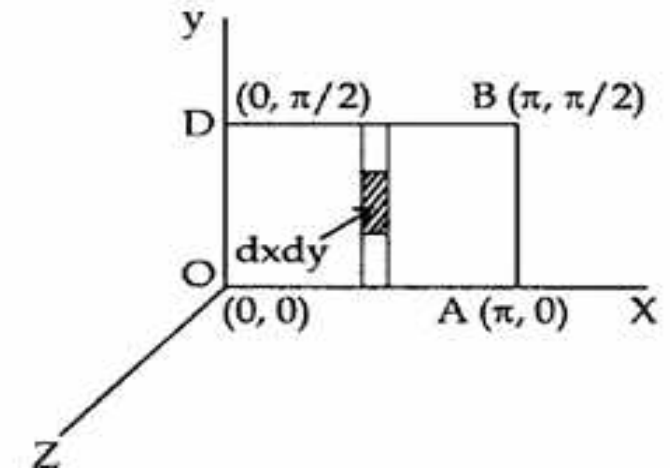
For this purpose, let us evaluate the given line integral directly.

$$\int_C [e^{-x} \sin y dx + e^{-x} \cos y dy]$$

$$= \int_{OA} [e^{-x} \sin y dx + e^{-x} \cos y dy] + \int_{AB} [e^{-x} \sin y dx + e^{-x} \cos y dy] +$$

$$\int_{BD} [e^{-x} \sin y dx + e^{-x} \cos y dy] + \int_{DO} [e^{-x} \sin y dx + e^{-x} \cos y dy]$$

Now along OA, $y = 0$	\Rightarrow	$dy = 0$
along AB, $x = \pi$	\Rightarrow	$dx = 0$
along BD, $y = \pi/2$	\Rightarrow	$dy = 0$
along DO, $x = 0$	\Rightarrow	$dx = 0$



Hence the given line integral

$$\begin{aligned} &= 0 + \int_0^{\pi/2} e^{-\pi} \cos y \, dy + \int_{\pi}^0 e^{-x} \, dx + \int_{\pi/2}^0 \cos y \, dy \\ &= e^{-\pi} [\sin y]_0^{\pi/2} + [-e^{-x}]_{\pi}^0 + [\sin y]_{\pi/2}^0 \\ &= e^{-\pi} - (1 - e^{-\pi}) + (-1) = 2(e^{-\pi} - 1) \end{aligned}$$

Hence Green's theorem is verified.

Example 2

State and verify Green's Theorem in the plane for $\oint (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the boundary of the region bounded by $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$.

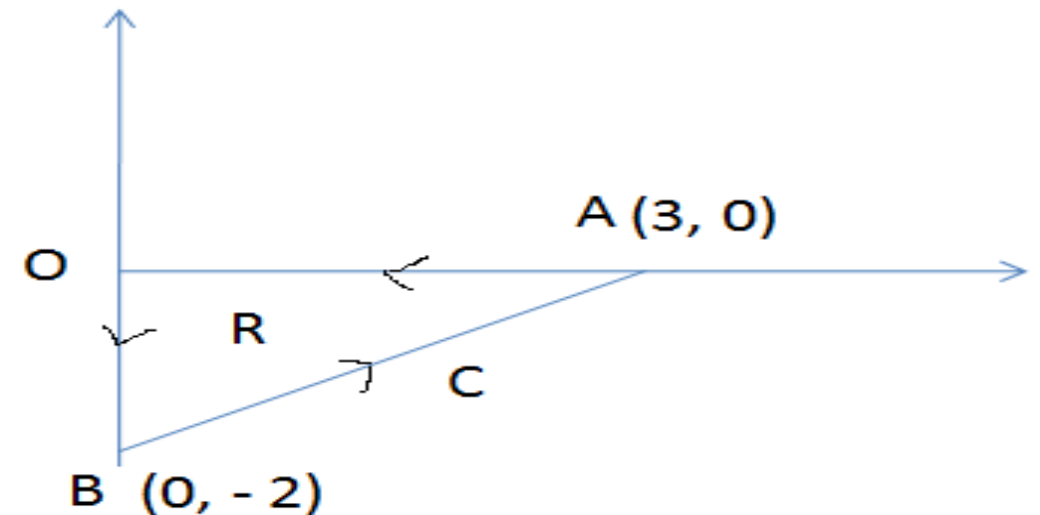
Solution

Here the closed curve C consists of straight lines OB , BA and AO , where coordinates of A and B are $(3, 0)$ and $(0, -2)$ respectively. Let R be the region bounded by C .

Then by Green's Theorem in plane, we have

$$\oint [(3x^2 - 8y^2) dx + (4y - 6xy) dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x} (4y - 6xy) - \frac{\partial}{\partial y} (3x^2 - 8y^2) \right] dx dy \quad \dots(1)$$



$$\begin{aligned}
 &= \iint_R (-6y + 16y) \, dx \, dy = \iint_R 10y \, dx \, dy \\
 &= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^0 y \, dy = 10 \int_0^3 dx \left[\frac{y^2}{2} \right]_{\frac{1}{3}(2x-6)}^0 = -\frac{5}{9} \int_0^3 dx (2x-6)^2 \\
 &= -\frac{5}{9} \left[\frac{(2x-6)^3}{3 \times 2} \right]_0^3 = -\frac{5}{54} (0 + 6)^3 = -\frac{5}{54} (216) = -20 \quad \dots(2)
 \end{aligned}$$

Now we evaluate L.H.S. of (1) along OB , BA and AO .

Along OB , $x = 0$, $dx = 0$ and y varies from 0 to -2 .

Along BA , $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2} dy$ and y varies from -2 to 0.

and along AO , $y = 0$, $dy = 0$ and x varies from 3 to 0.

$$\text{L.H.S. of (1)} = \oint [(3x^2 - 8y^2) dx + (4y - 6xy) dy]$$

$$\begin{aligned}
 &= \int_{OB} [(3x^2 - 8y^2) dx + (4y - 6xy) dy] + \int_{BA} [(3x^2 - 8y^2) dx + (4x - 6xy) dy] \\
 &\quad + \int_{AO} [(3x^2 - 8y^2) dx + (4y - 6xy) dy] \\
 &= \int_0^{-2} 4y dy + \int_{-2}^0 \left[\frac{3}{4} (6 + 3y)^2 - 8y^2 \right] \left(\frac{3}{2} dy \right) + [4y - 3(6 + 3y)y] dy + \int_3^0 3x^2 dx \\
 &= [2y^2]_0^{-2} + \int_{-2}^0 \left[\frac{9}{8} (6 + 3y)^2 - 12y^2 + 4y - 18y - 9y^2 \right] dy + (x^3)_3^0 \\
 &= 2[4] + \int_{-2}^0 \left[\frac{9}{8} (6 + 3y)^2 - 21y^2 - 14y \right] dy + (0 - 27) \\
 &= 8 + \left[\frac{9}{8} \frac{(6 + 3y)^3}{3 \times 3} - 7y^3 - 7y^2 \right]_{-2}^0 - 27 = -19 + \left[\frac{216}{8} + 7(-2)^3 + 7(-2)^2 \right] \\
 &= -19 + 27 - 56 + 28 = -20 \quad \dots(3)
 \end{aligned}$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

Q.2: Verify Green's theorem in plane for: $\oint_C (x^2 - 2xy) dx + (x^2y + 3) dy$,
 where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$.

Soln: Using Green's theorem

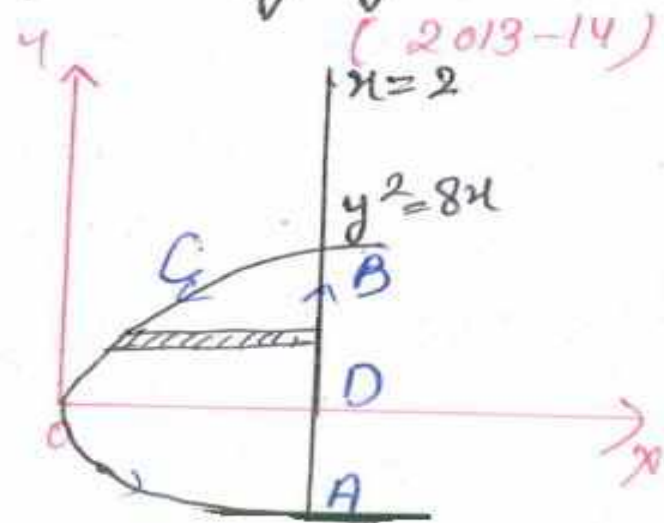
$$\oint_C (x^2 - 2xy) dx + (x^2y + 3) dy$$

$$= \iint_R \left[\frac{\partial}{\partial x} (x^2y + 3) - \frac{\partial}{\partial y} (x^2 - 2xy) \right] dx dy$$

$$= \iint_R (2xy + 2x) dx dy = \int_{y=-4}^4 \int_{x=y^2/8}^2 (2xy + 2x) dx dy$$

$$= \int_{y=-4}^4 \left[x^2y + x^2 \right]_{x=y^2/8}^2 dy = \int_{y=-4}^4 \left[4y + 4 - \frac{y^5}{64} - \frac{y^4}{64} \right] dy$$

$$= \left[2y^2 + 4y - \frac{y^6}{64 \times 6} - \frac{y^5}{64 \times 5} \right]_{-4}^4 = \frac{128}{5}$$



Verification of the Green's theorem:

$$\oint_C (x^2 - 2xy) dx + (x^2y + 3) dy = \left[\int_{BOA} [(x^2 - 2xy) dx + (x^2y + 3) dy] \right]$$

$$+ \int_{AOB} [(x^2 - 2xy) dx + (x^2y + 3) dy]$$

Along BOA, $y^2 = 8x$, $dx = \frac{y}{4} dy$

Along AOB, $x = 2$, $dx = 0$

$$= \int_{y=4}^{-4} \left[\left(\frac{y^4}{64} - \frac{y^3}{4} \right) \frac{y}{4} dy + \left(\frac{y^5}{64} + 3 \right) dy \right]$$

$$+ \int_{y=-4}^4 [(4 - 4y) \cdot 0 + (4y + 3) dy]$$

$$= \left[\frac{y^6}{64 \times 24} - \frac{y^5}{16 \times 5} + \frac{y^6}{64 \times 6} + 3y \right]_{-4}^4 + \left[2y^2 + 3y \right]_{-4}^4$$

$$= \frac{128}{5} - 24 + 24 = \frac{128}{5}$$

Verified

Q.3: Verify the Green's theorem to evaluate the line integral $\int_C (2y^2 dx + 3x dy)$, where C is the boundary of the closed region bounded by $y=x$ and $y=x^2$. (2015-16)

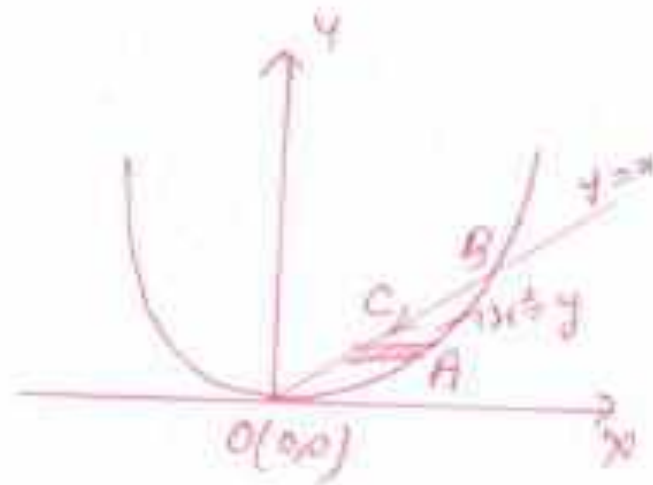
Solⁿ: Using Green's theorem

$$\int_C 2y^2 dx + 3x dy = \iint \left[\frac{\partial}{\partial x} (3x) - \frac{\partial}{\partial y} (2y^2) \right] dx dy$$

$$= \iint (3 - 4y) dx dy = \int_0^1 \int_{x=y}^{\sqrt{y}} (3 - 4y) dx dy$$

$$= \int_0^1 [3x - 4xy]_y^{\sqrt{y}} dy = \int_0^1 (3\sqrt{y} - 4y^{3/2} - 3y + 4y^2) dy$$

$$= \left[3 \times \frac{2}{3} y^{3/2} - 4 \times \frac{2}{5} y^{5/2} - \frac{3y^2}{2} + \frac{4y^3}{3} \right]_0^1 = \frac{6}{3} - \frac{8}{5} - \frac{3}{2} + \frac{4}{3} = \frac{7}{30}$$



Verification of the Green's theorem:

$$\int_C 2y^2 dx + 3x dy = \int_{OAB} (2y^2 dx + 3x dy) + \int_{BCO} (2y^2 dx + 3x dy)$$

Along OAB, $y = x^2$, $dy = 2x dx$

Along BCO, $y = x$, $dy = dx$

$$= \int_0^1 [2x^4 + 3x(2x)] dx + \int_1^0 (2x^2 + 3x) dx$$

$$= \left[\frac{2x^5}{5} + 2x^3 \right]_0^1 + \left[\frac{2x^3}{3} + \frac{3x^2}{2} \right]_1^0$$

$$= \left(\frac{2}{5} + 2 \right) - \left(\frac{2}{3} + \frac{3}{2} \right) = \frac{12}{5} - \frac{13}{6} = \frac{72 - 65}{30} = \frac{7}{30} \quad \text{Verified}$$

Q.46: Verify Green's theorem, evaluate $\int_C (x^2+xy) dx + (x^2+y^2) dy$, where C square formed by lines $x = \pm 1, y = \pm 1$. (2017-18)

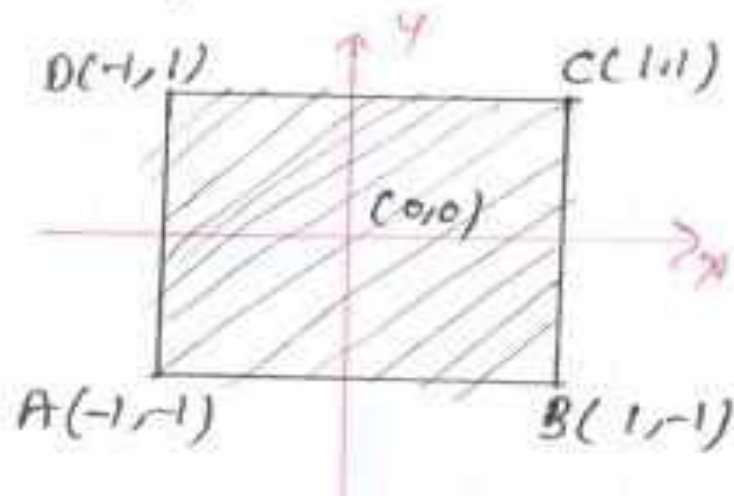
Solⁿ: By Green's theorem, we have

$$\int_C (x^2+xy) dx + (x^2+y^2) dy = \iint_R \left[\frac{\partial}{\partial x} (x^2+y^2) - \frac{\partial}{\partial y} (x^2+xy) \right] dx dy$$

$$= \iint_R (2x-x) dx dy = \int_{x=-1}^1 \int_{y=-1}^1 x dx dy = \int_{-1}^1 x (y)_{-1}^1 dx = \int_{-1}^1 2x dx = 0$$

Verification of the Green's theorem:

$$\int_C [(x^2+xy) dx + (x^2+y^2) dy] = \int_{AB} [(x^2+xy) dx + (x^2+y^2) dy] +$$



$$\int_{BC} [(x^2 + xy) dx + (x^2 + y^2) dy] + \int_{CD} [(x^2 + xy) dx + (x^2 + y^2) dy]$$

$$+ \int_{DA} [(x^2 + xy) dx + (x^2 + y^2) dy]$$

Now along AB, $y = -1$ and $dy = 0$

$$\therefore \int_{AB} [(x^2 + xy) dx + (x^2 + y^2) dy] = \int_{-1}^1 (x^2 - x) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_{-1}^1$$

$$= \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2} \right] = \frac{2}{3}$$

Along BC, $x = 1$ and $dx = 0$

$$\therefore \int_{BC} [(x^2 + xy) dx + (x^2 + y^2) dy] = \int_{-1}^1 (1 + y^2) dy = \left[y + \frac{y^3}{3} \right]_{-1}^1 = \frac{8}{3}$$

Along CD, $y=1$ and $dy=0$

$$\therefore \int_{CD} [(x^2+xy) dx + (x^2+y^2) dy] = \int_{-1}^1 (x^2+x) dx = \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^1$$

$$= \left[\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} \right] = 0$$

Along DA, $x=-1$ and $dx=0$

$$\therefore \int_{DA} [(x^2+xy) dx + (x^2+y^2) dy] = \int_1^{-1} (1+y^2) dy = \left[y + \frac{y^3}{3} \right]_1^{-1}$$

$$= \left[-1 - \frac{1}{3} - 1 - \frac{1}{3} \right] = -\frac{8}{3}$$

$$\Rightarrow \int_C [(x^2+xy) dx + (x^2+y^2) dy] = \frac{2}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3} = 0 \quad \text{Verified}$$

Practice Questions

1 Verify Green's Theorem in plane for $\int_C (x^2 + 2xy) dx + (y^2 + x^3y) dy$, where c is a square with the vertices $P (0, 0)$, $Q (1, 0)$, $R (1, 1)$ and $S (0, 1)$. Ans. $-\frac{1}{2}$

2 Verify Green's Theorem for $\int_C [(xy + y^2) dx + x^2 dy]$ where C is the boundary by $y = x$ and $y = x^2$.

3 Verify Green's Theorem for $\int_C (x^2 - 2xy) dx + (x^2y + 3) dy$ around the boundary c of the region $y^2 = 8x$ and $x = 2$.

4 Verify the Green's Theorem to evaluate the line integral $\int_C (2y^2 dx + 3x dy)$, where c is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

Ans. $\frac{27}{4}$

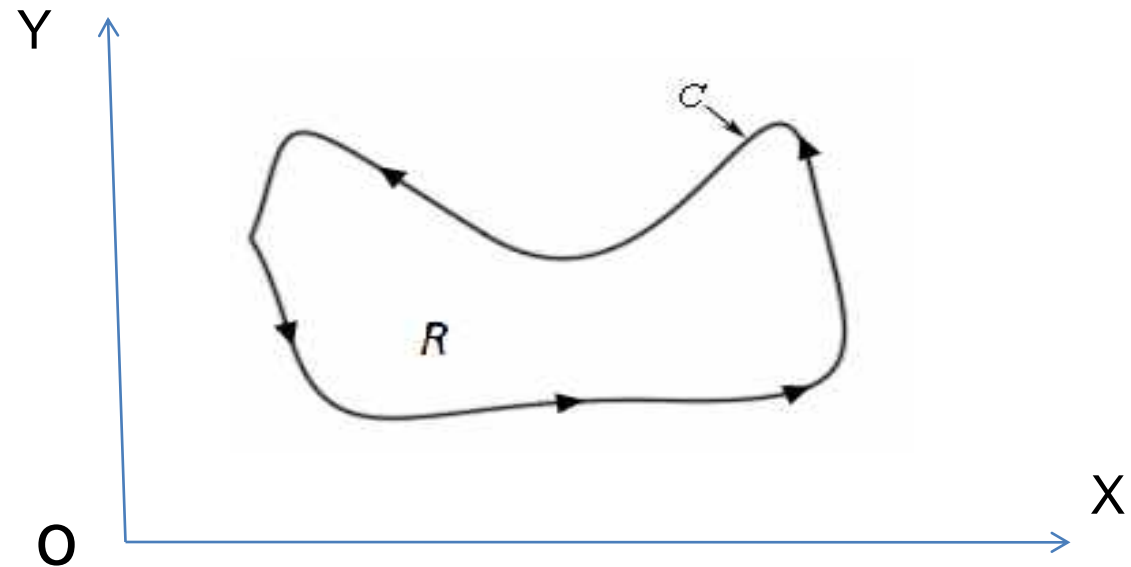
Lecture 41 (II)

Green's Theorem and its Applications - II

Green Theorem

If $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in $x - y$ plane, then

$$\oint_C (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy$$



Area of Plane Region by Green Theorem

We know that

$$\int_C M dx + N dy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \quad \dots(1)$$

On putting

$$N = x \left(\frac{\partial N}{\partial x} = 1 \right) \text{ and } M = -y \left(\frac{\partial M}{\partial y} = 1 \right) \text{ in (1), we get}$$

$$\int_C -y dx + x dy = \iint_A [1 - (-1)] dx dy = 2 \iint dx dy = 2 A$$

$$\text{Area} = \frac{1}{2} \int_C (x dy - y dx)$$

Example 3

Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$$y = x, y = \frac{1}{x}, y = \frac{x}{4}$$

Solution

By Green's Theorem Area A of the region bounded by a closed curve C is given by

$$A = \frac{1}{2} \oint_C (x dy - y dx)$$

Here, C consists of the curves $C_1 : y = \frac{x}{4}$, $C_2 : y = \frac{1}{x}$

and $C_3 : y = x$ So

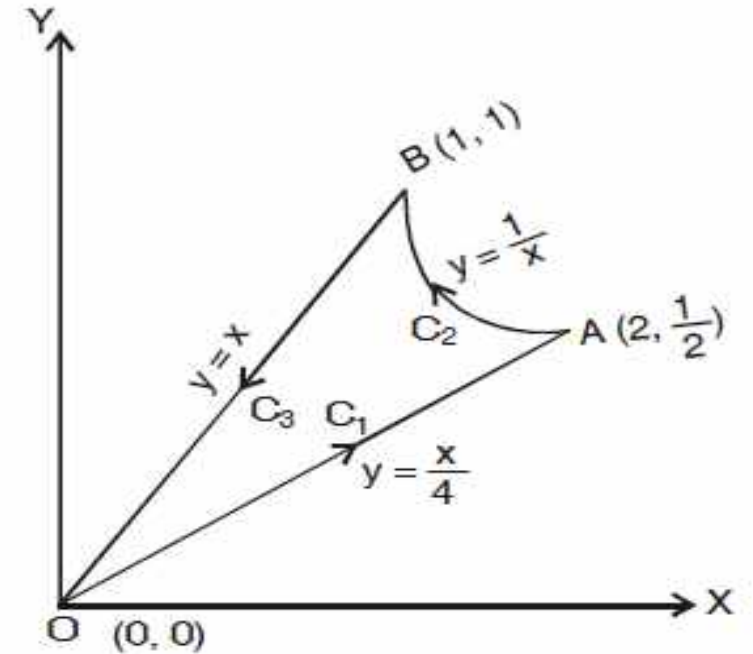
$$\left[A = \frac{1}{2} \oint_C = \frac{1}{2} \left[\int_{C_1} + \int_{C_2} + \int_{C_3} \right] = \frac{1}{2} (I_1 + I_2 + I_3) \right]$$

Along $C_1 : y = \frac{x}{4}, dy = \frac{1}{4} dx, x : 0 \text{ to } 2$

$$I_1 = \int_{C_1} (x dy - y dx) = \int_{C_1} \left(x \frac{1}{4} dx - \frac{x}{4} dx \right) = 0$$

Along $C_2 : y = \frac{1}{x}, dy = -\frac{1}{x^2} dx, x : 2 \text{ to } 1$

$$I_2 = \int_{C_2} (x dy - y dx) = \int_2^1 \left[x \left(-\frac{1}{x^2} \right) dx - \frac{1}{2} dx \right] = [-2 \log x]_2^1 = 2 \log 2$$



Example 1

Evaluate by Green's theorem $\int_C [e^{-x} \sin y dx + e^{-x} \cos y dy]$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi, \pi/2)$, $(0, \pi/2)$ and hence verify Green's theorem.

Solution

By Green's theorem we have

$$\int_C (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Comparing the given integral

$$M = e^{-x} \sin y$$

and

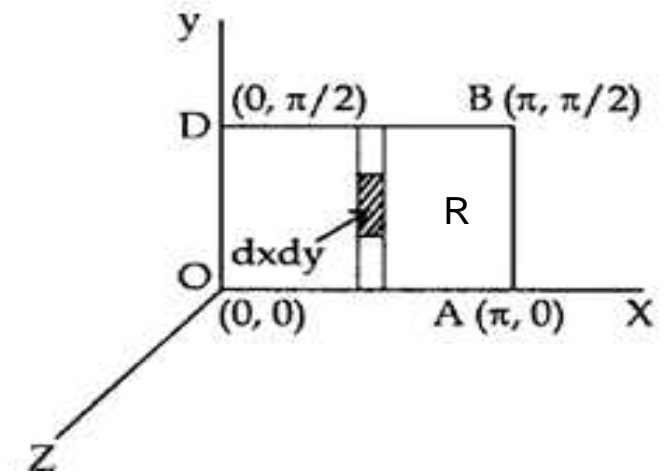
$$N = e^{-x} \cos y$$

$$\text{Then } \frac{\partial M}{\partial y} = e^{-x} \cos y$$

and

$$\frac{\partial N}{\partial x} = -e^{-x} \cos y$$

Hence by Green's theorem



$$\begin{aligned}
 \int_C [e^{-x} \sin y dx + e^{-x} \cos y dy] &= \iint_R (-e^{-x} \cos y - e^{-x} \cos y) dx dy \\
 &= -2 \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} e^{-x} \cos y dx dy \\
 &= -2 \left[-e^{-x} \right] \left[\sin y \right]_0^{\pi/2} \\
 &= 2(e^{-\pi} - 1)(1) \\
 &= 2(e^{-\pi} - 1)
 \end{aligned}$$

Evaluation of line integral:

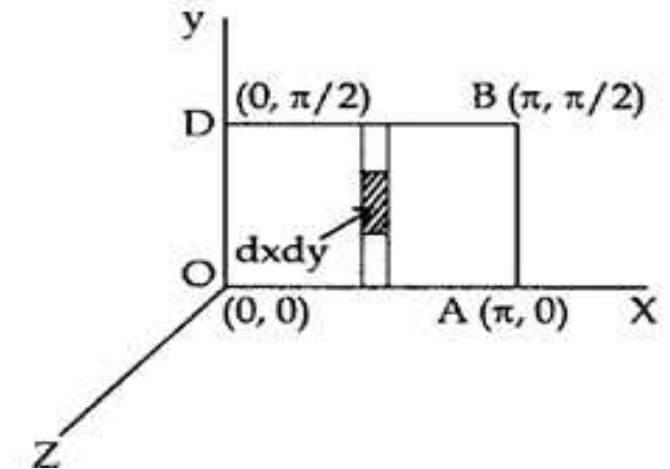
For this purpose, let us evaluate the given line integral directly.

$$\int_C [e^{-x} \sin y dx + e^{-x} \cos y dy]$$

$$= \int_{OA} [e^{-x} \sin y dx + e^{-x} \cos y dy] + \int_{AB} [e^{-x} \sin y dx + e^{-x} \cos y dy] +$$

$$\int_{BD} [e^{-x} \sin y dx + e^{-x} \cos y dy] + \int_{DO} [e^{-x} \sin y dx + e^{-x} \cos y dy]$$

Now along OA, $y = 0$	\Rightarrow	$dy = 0$
along AB, $x = \pi$	\Rightarrow	$dx = 0$
along BD, $y = \pi/2$	\Rightarrow	$dy = 0$
along DO, $x = 0$	\Rightarrow	$dx = 0$



Hence the given line integral

$$\begin{aligned} &= 0 + \int_0^{\pi/2} e^{-\pi} \cos y \, dy + \int_{\pi}^0 e^{-x} \, dx + \int_{\pi/2}^0 \cos y \, dy \\ &= e^{-\pi} [\sin y]_0^{\pi/2} + [-e^{-x}]_{\pi}^0 + [\sin y]_{\pi/2}^0 \\ &= e^{-\pi} - (1 - e^{-\pi}) + (-1) = 2(e^{-\pi} - 1) \end{aligned}$$

Hence Green's theorem is verified.

Example 2

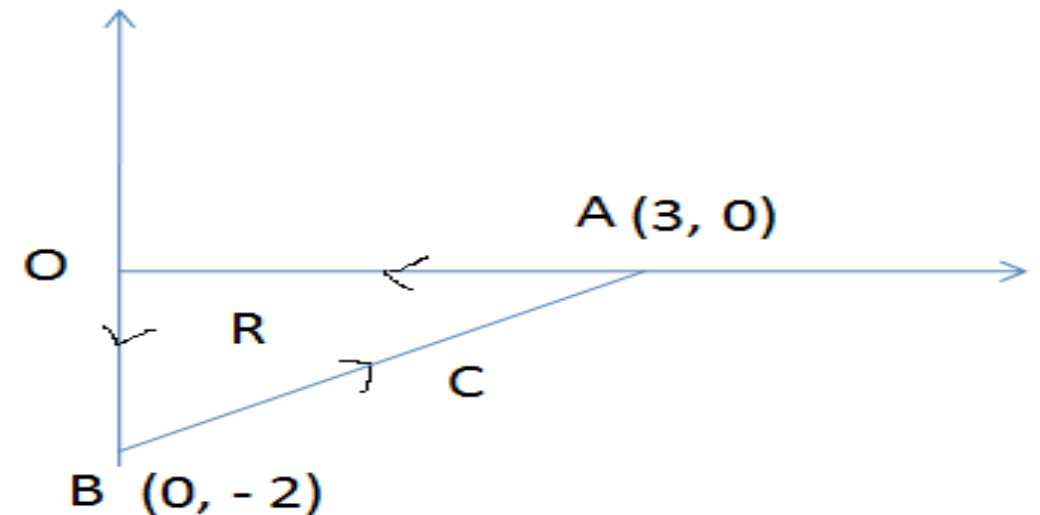
State and verify Green's Theorem in the plane for $\oint (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the boundary of the region bounded by $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$.

Solution

Here the closed curve C consists of straight lines OB , BA and AO , where coordinates of A and B are $(3, 0)$ and $(0, -2)$ respectively. Let R be the region bounded by C .

Then by Green's Theorem in plane, we have

$$\begin{aligned} & \oint [(3x^2 - 8y^2) dx + (4y - 6xy) dy] \\ &= \iint_R \left[\frac{\partial}{\partial x} (4y - 6xy) - \frac{\partial}{\partial y} (3x^2 - 8y^2) \right] dx dy \quad \dots(1) \end{aligned}$$



$$\begin{aligned}
 &= \iint_R (-6y + 16y) \, dx \, dy = \iint_R 10y \, dx \, dy \\
 &= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^0 y \, dy = 10 \int_0^3 dx \left[\frac{y^2}{2} \right]_{\frac{1}{3}(2x-6)}^0 = -\frac{5}{9} \int_0^3 dx (2x-6)^2 \\
 &= -\frac{5}{9} \left[\frac{(2x-6)^3}{3 \times 2} \right]_0^3 = -\frac{5}{54} (0 + 6)^3 = -\frac{5}{54} (216) = -20 \quad \dots(2)
 \end{aligned}$$

Now we evaluate L.H.S. of (1) along OB , BA and AO .

Along OB , $x = 0$, $dx = 0$ and y varies from 0 to -2 .

Along BA , $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2} dy$ and y varies from -2 to 0.

and along AO , $y = 0$, $dy = 0$ and x varies from 3 to 0.

$$\text{L.H.S. of (1)} = \oint [(3x^2 - 8y^2) dx + (4y - 6xy) dy]$$

$$\begin{aligned}
 &= \int_{OB} [(3x^2 - 8y^2) dx + (4y - 6xy) dy] + \int_{BA} [(3x^2 - 8y^2) dx + (4x - 6xy) dy] \\
 &\quad + \int_{AO} [(3x^2 - 8y^2) dx + (4y - 6xy) dy] \\
 &= \int_0^{-2} 4y dy + \int_{-2}^0 \left[\frac{3}{4} (6 + 3y)^2 - 8y^2 \right] \left(\frac{3}{2} dy \right) + [4y - 3(6 + 3y)y] dy + \int_3^0 3x^2 dx \\
 &= [2y^2]_0^{-2} + \int_{-2}^0 \left[\frac{9}{8} (6 + 3y)^2 - 12y^2 + 4y - 18y - 9y^2 \right] dy + (x^3)_3^0 \\
 &= 2[4] + \int_{-2}^0 \left[\frac{9}{8} (6 + 3y)^2 - 21y^2 - 14y \right] dy + (0 - 27) \\
 &= 8 + \left[\frac{9}{8} \frac{(6 + 3y)^3}{3 \times 3} - 7y^3 - 7y^2 \right]_{-2}^0 - 27 = -19 + \left[\frac{216}{8} + 7(-2)^3 + 7(-2)^2 \right] \\
 &= -19 + 27 - 56 + 28 = -20 \quad \dots(3)
 \end{aligned}$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

Practice Questions

- 1 Apply Green's Theorem to evaluate $\int_c [(y - \sin x) dy + \cos x dx]$, where c is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$. Ans. $-\frac{\pi^2 + 8}{4\pi}$
- 2 Use Green's Theorem in a plane to evaluate the integral $\int_c [(2x^2 - y^2) dx + (x^2 + y^2) dy]$, where c is the boundary in the xy -plane of the area enclosed by the x -axis and the semi-circle $x^2 + y^2 = 1$ in the upper half xy -plane. Ans. $\frac{4}{3}$
- 3 Green's Theorem, evaluate the line integral $\int_c e^{-x} (\cos y dx - \sin y dy)$, where c is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$ and $(0, \frac{\pi}{2})$. Ans. $2(1 - e^{-\pi})$
- 4 Use Green's theorem to evaluate $\int_C (x^2 + xy) dx + (x^2 + y^2) dy$ where C is the square formed by the lines $y = \pm 1$, $x = \pm 1$.

Ans: 0

Practice Questions

1 Verify Green's Theorem in plane for $\int_C (x^2 + 2xy) dx + (y^2 + x^3y) dy$, where c is a square with the vertices $P (0, 0)$, $Q (1, 0)$, $R (1, 1)$ and $S (0, 1)$. Ans. $-\frac{1}{2}$

2 Verify Green's Theorem for $\int_C [(xy + y^2) dx + x^2 dy]$ where C is the boundary by $y = x$ and $y = x^2$.

3 Verify Green's Theorem for $\int_C (x^2 - 2xy) dx + (x^2y + 3) dy$ around the boundary c of the region $y^2 = 8x$ and $x = 2$.

4 Verify the Green's Theorem to evaluate the line integral $\int_C (2y^2 dx + 3x dy)$, where c is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

Ans. $\frac{27}{4}$

Lecture 42(I)

Stoke's Theorem and It's Applications - I

Stoke's Theorem

Surface integral of the component of $\text{curl } \vec{F}$ along the normal to the surface S , taken over the surface S bounded by curve C is equal to the line integral of the vector point function

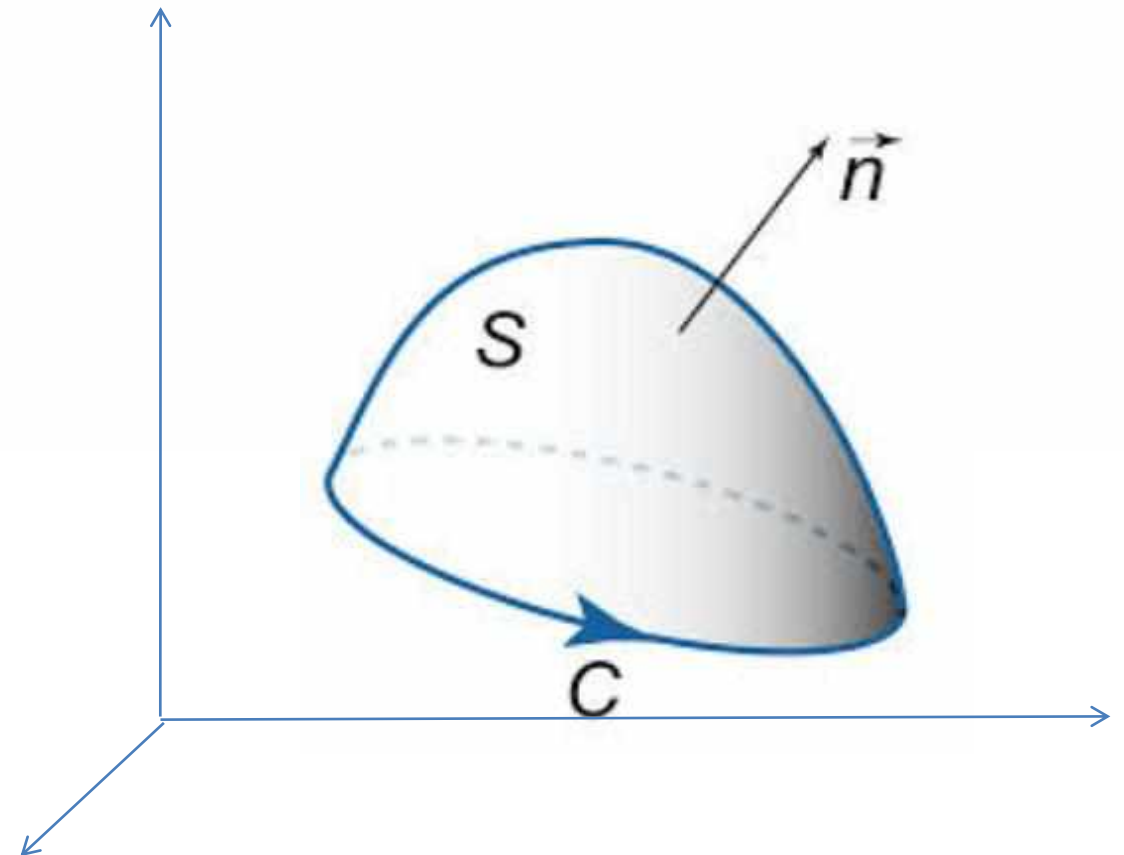
\vec{F} taken along the closed curve C .

Mathematically

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds$$

where \hat{n} is a unit external normal to the surface.

Stokes' Theorem relates a surface integral over an open surface S to a line integral around the boundary curve of S (a space curve).



Example 1

Using Stoke's theorem or otherwise, evaluate

$$\int_c [(2x - y) dx - yz^2 dy - y^2 z dz]$$

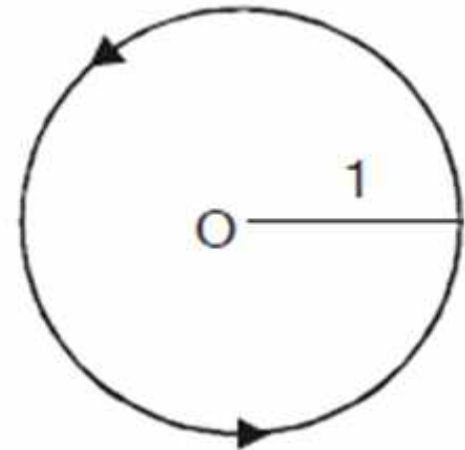
where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

Solution

$$\int_c [(2x - y) dx - yz^2 dy - y^2 z dz]$$

$$= \int_c [(2x - y) \hat{i} - yz^2 \hat{j} - y^2 z \hat{k}] \cdot (\hat{i} dx + \hat{j} dy + \hat{k} dz)$$

By Stoke's theorem $\oint \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} ds$



$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\hat{i} - (0 - 0)\hat{j} + (0 + 1)\hat{k} = \hat{k}$$

Putting the value of curl \vec{F} in (1), we get

$$= \iint \hat{k} \cdot \hat{n} \, ds = \iint \hat{k} \cdot \hat{n} \frac{dx \, dy}{\hat{n} \cdot \hat{k}} = \iint dx \, dy = \text{Area of the circle} = \pi \quad \left[\because ds = \frac{dx \, dy}{(\hat{n} \cdot \hat{k})} \right]$$

Example 2

Verify Stoke's Theorem for the function

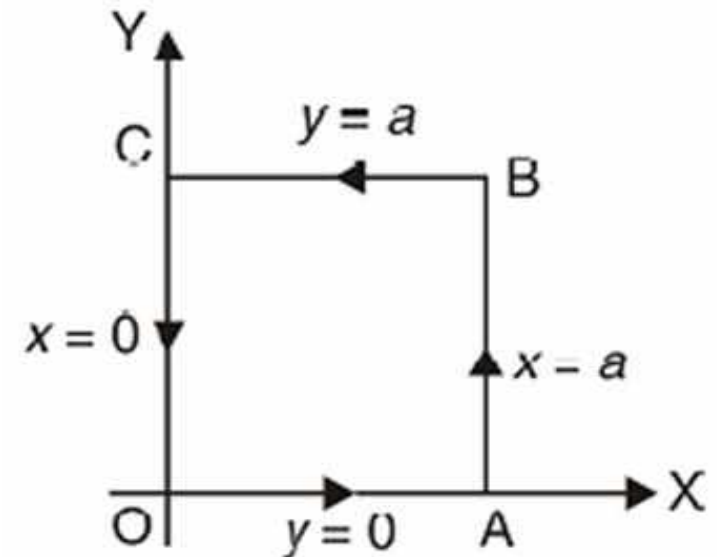
$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane $z = 0$ and bounded by the lines
 $x = 0, y = 0, x = a, y = a$.

Solution

We have, $\vec{F} = x^2\hat{i} - xy\hat{j}$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix} \\ &= (0 - 0)\hat{i} - (0 - 0)\hat{j} + (-y - 0)\hat{k} = -y\hat{k} \end{aligned}$$



($\hat{n} \perp$ to xy plane *i.e.* \hat{k})

$$\begin{aligned} \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds &= \iint_S (-yk) \cdot k \, dx \, dy \\ &= \int_0^a dx \int_0^a -y \, dy = \int_0^a dx \left[-\frac{y^2}{2} \right]_0^a = -\frac{a^2}{2} (x)_0^a = -\frac{a^3}{2} \end{aligned} \quad \dots(1)$$

To obtain line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (x^2 \hat{i} - xy \hat{j}) \cdot (\hat{i} \, dx + \hat{j} \, dy) = \int_C (x^2 \, dx - xy \, dy)$$

where c is the path $OABCO$ as shown in the figure.

$$\text{Also, } \int_C \vec{F} \cdot d\vec{r} = \int_{OABCO} \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r} \quad \dots(2)$$

Along $\sim OA$, $y = 0$, $dy = 0$

$$\begin{aligned} \int_{OA} \vec{F} \cdot \vec{dr} &= \int_{OA} (x^2 dx - xy dy) \\ &= \int_0^a x^2 dx = \left[\frac{x^3}{3} \right]_0^a = \frac{a^3}{3} \end{aligned}$$

Along AB , $x = a$, $dx = 0$

$$\begin{aligned} \int_{AB} \vec{F} \cdot \vec{dr} &= \int_{AB} (x^2 dx - xy dy) \\ &= \int_0^a -a y dy = -a \left[\frac{y^2}{2} \right]_0^a = -\frac{a^3}{2} \end{aligned}$$

line	Eq. of line		Lower limit	Upper limit
OA	$y = 0$	$dy = 0$	$x = 0$	$x = a$
AB	$x = a$	$dx = 0$	$y = 0$	$y = a$
BC	$y = a$	$dy = 0$	$x = a$	$x = 0$
CO	$x = 0$	$dx = 0$	$y = a$	$y = 0$

Along BC , $y = a, dy = 0$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_{BC} (x^2 dx - xy dy) = \int_a^0 x^2 dx = \left[\frac{x^3}{3} \right]_a^0 = -\frac{a^3}{3}$$

Along CO , $x = 0, dx = 0$

$$\int_{CO} \vec{F} \cdot \vec{dr} = \int_{CO} (x^2 dx - xy dy) = 0$$

Putting the values of these integrals in (2), we have

$$\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} - \frac{a^3}{2} - \frac{a^3}{3} + 0 = -\frac{a^3}{2} \quad \dots(3)$$

From (1) and (3),
$$\iiint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} ds = \int_C \vec{F} \cdot \vec{dr}$$

Hence, Stoke's Theorem is verified.

Ans.

Example 3

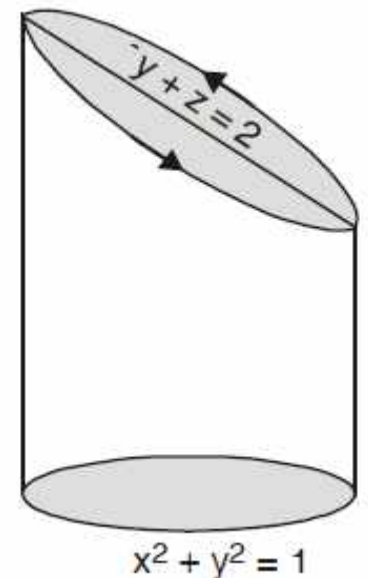
Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} ds = \iint_S \text{curl} (-y^2\hat{i} + x\hat{j} + z^2\hat{k}) \cdot \hat{n} ds \quad \dots(1)$$

$$F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+2y) = (1+2y)\hat{k}$$



$$\text{Normal vector} = \nabla \vec{F}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (y + z - 2) = \hat{j} + \hat{k}$$

$$\text{Unit normal vector } \hat{n} = \frac{\hat{j} + \hat{k}}{\sqrt{2}}$$

$$ds = \frac{dx dy}{\hat{n} \cdot \hat{k}}$$

On putting the values of curl \vec{F} , \hat{n} and ds in (1), we get

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (1 + 2y) \hat{k} \cdot \frac{\hat{j} + \hat{k}}{\sqrt{2}} \frac{dx dy}{\left(\frac{\hat{j} + \hat{k}}{\sqrt{2}} \right) \cdot \hat{k}}$$

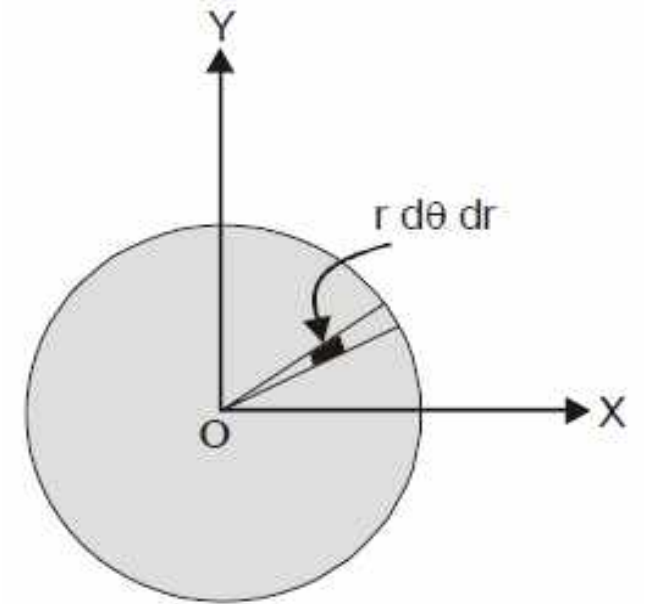
$$= \iint \frac{1 + 2y}{\sqrt{2}} \frac{dx dy}{\frac{1}{\sqrt{2}}} = \iint (1 + 2y) dx dy$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r \, d\theta \, dr$$

$$= \int_0^{2\pi} \int_0^1 (r + 2r^2 \sin \theta) \, d\theta \, dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3} \sin \theta \right]_0^1 = \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3} \sin \theta \right] d\theta$$

$$= \left[\frac{\theta}{2} - \frac{2}{3} \cos \theta \right]_0^{2\pi} = \left(\pi - \frac{2}{3} - 0 + \frac{2}{3} \right) = \pi$$



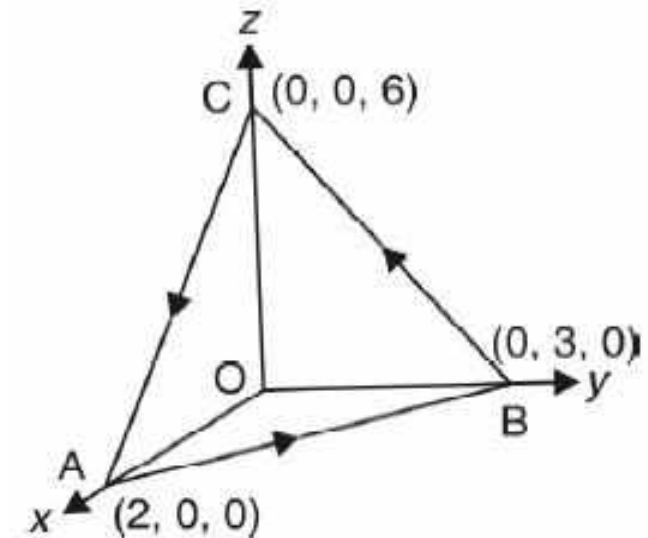
Example 3

Verify Stoke's Theorem for $\vec{F} = (x + y) \hat{i} + (2x - z) \hat{j} + (y + z) \hat{k}$ for the surface of a triangular lamina with vertices $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$.

Solution

Here the path of integration c consists of the straight lines AB, BC, CA where the co-ordinates of A, B, C are $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$ respectively. Let S be the plane surface of triangle ABC bounded by C . Let \hat{n} be unit normal vector to surface S . Then by Stoke's Theorem, we must have

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_s \text{curl } \vec{F} \cdot \hat{n} ds \quad \dots(1)$$



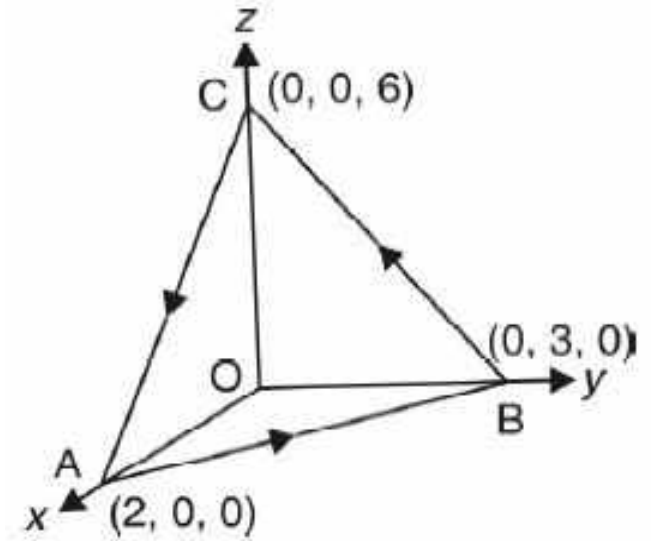
$$\text{L.H.S. of (1)} = \int_{ABC} \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CA} \vec{F} \cdot d\vec{r}$$

Along line AB , $z = 0$, equation of AB is $\frac{x}{2} + \frac{y}{3} = 1$

$$\Rightarrow y = \frac{3}{2}(2 - x), \quad dy = -\frac{3}{2}dx$$

At A , $x = 2$, At B , $x = 0$, $\vec{r} = x\hat{i} + y\hat{j}$

$$\begin{aligned} \int_{AB} \vec{F} \cdot d\vec{r} &= \int_{AB} [(x + y)\hat{i} + 2x\hat{j} + y\hat{k}] \cdot (\hat{i}dx + \hat{j}dy) \\ &= \int_{AB} (x + y) dx + 2x dy \\ &= \int_{AB} \left(x + 3 - \frac{3x}{2} \right) dx + 2x \left(-\frac{3}{2} dx \right) \\ &= \int_2^0 \left(-\frac{7x}{2} + 3 \right) dx = \left(-\frac{7x^2}{4} + 3x \right)_2^0 \\ &= (7 - 6) = +1 \end{aligned}$$

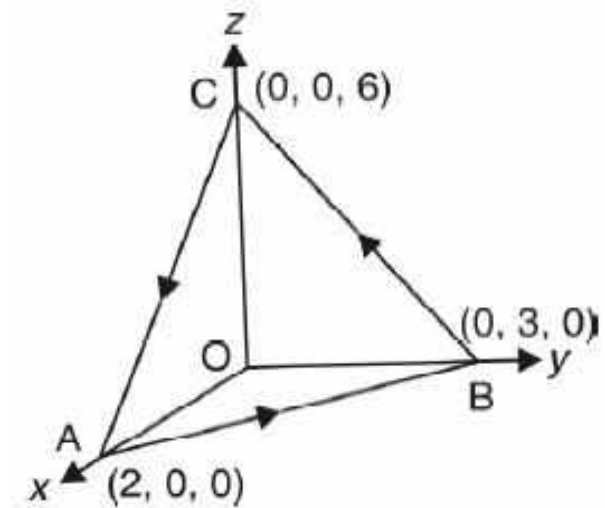


line	Eq. of line		Lower limit	Upper limit
AB	$\frac{x}{2} + \frac{y}{3} = 1$ $z = 0$	$dy = -\frac{3}{2} dx$	At A $x = 2$	At B $x = 0$

Along line BC , $x = 0$, Equation of BC is $\frac{y}{3} + \frac{z}{6} = 1$ or $z = 6 - 2y$, $dz = -2dy$

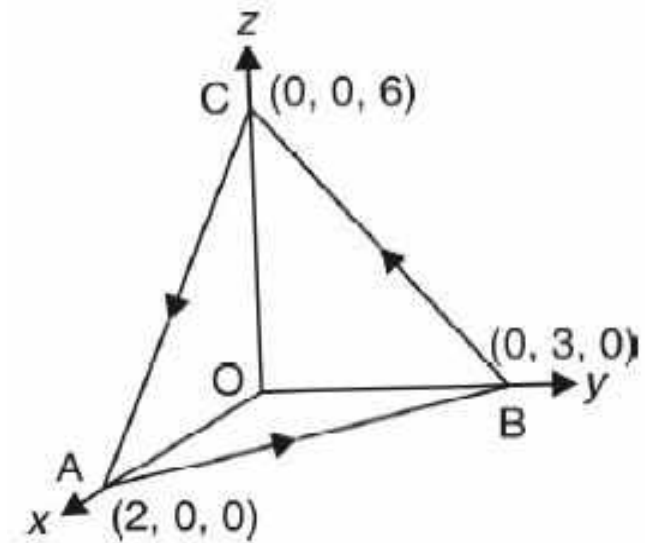
At B , $y = 3$, At C , $y = 0$, $\vec{r} = y\hat{j} + z\hat{k}$

$$\begin{aligned} \int_{BC} \vec{F} \cdot d\vec{r} &= \int_{BC} [yi + zj + (y+z)k] \cdot (jdy + kdz) = \int_{BC} -zdy + (y+z) dz \\ &= \int_3^0 (-6 + 2y) dy + (y + 6 - 2y) (-2dy) \\ &= \int_3^0 (4y - 18) dy = (2y^2 - 18y)_3^0 = 36 \end{aligned}$$



line	Eq. of line		Lower limit	Upper limit
BC	$\frac{y}{3} + \frac{z}{6} = 1$ $x = 0$	$dz = -2dy$	At B $y = 3$	At C $y = 0$

line	Eq. of line		Lower limit	Upper limit
CA	$\frac{x}{2} + \frac{z}{6} = 1$ $y = 0$	$dz = -3dx$	At C $x = 0$	At A $x = 2$



Along line CA, $y = 0$, Eq. of CA, $\frac{x}{2} + \frac{z}{6} = 1$ or $z = 6 - 3x$, $dz = -3dx$

At C, $x = 0$, at A, $x = 2$, $\vec{F} = x\hat{i} + z\hat{k}$

$$\begin{aligned} \int_{CA} \vec{F} \cdot d\vec{r} &= \int_{CA} [x\hat{i} + (2x - z)\hat{j} + z\hat{k}] \cdot [dx\hat{i} + dz\hat{k}] = \int_{CA} (xdx + zdz) \\ &= \int_0^2 xdx + (6 - 3x)(-3dx) = \int_0^2 (10x - 18) dx = [5x^2 - 18x]_0^2 = -16 \end{aligned}$$

Lecture 42(II)

Stoke's Theorem and It's Applications - II

Q.1: Evaluate $\oint_C \vec{F} \cdot d\vec{s}$ by Stoke's theorem, where:

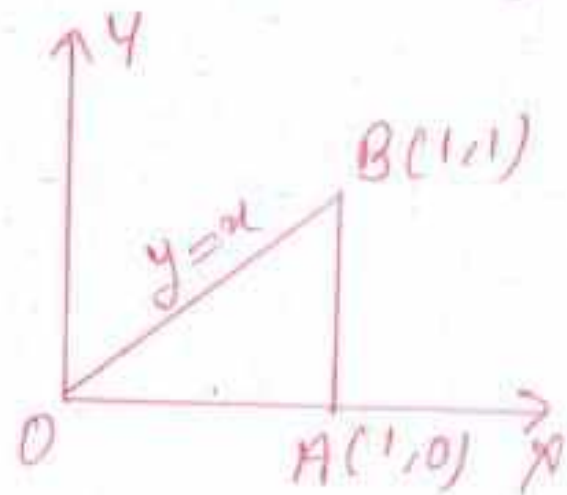
$\vec{F} = y^2 \hat{i} + x^2 \hat{j} - (x+z) \hat{k}$ and C is the boundary of triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$. (2013-14)

Solⁿ: Since z -coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy -plane and $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$

$$\therefore \text{curl } \vec{F} \cdot \hat{n} = [\hat{j} + 2(x-y)\hat{k}] \cdot \hat{k} = 2(x-y)$$

The equation of line OB is $y=x$.



By Stoke's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds$$

$$= \int_0^1 \int_0^x 2(x-y) \, dy \, dx = \int_0^1 2 \left[xy - \frac{y^2}{2} \right]_0^x \, dx$$

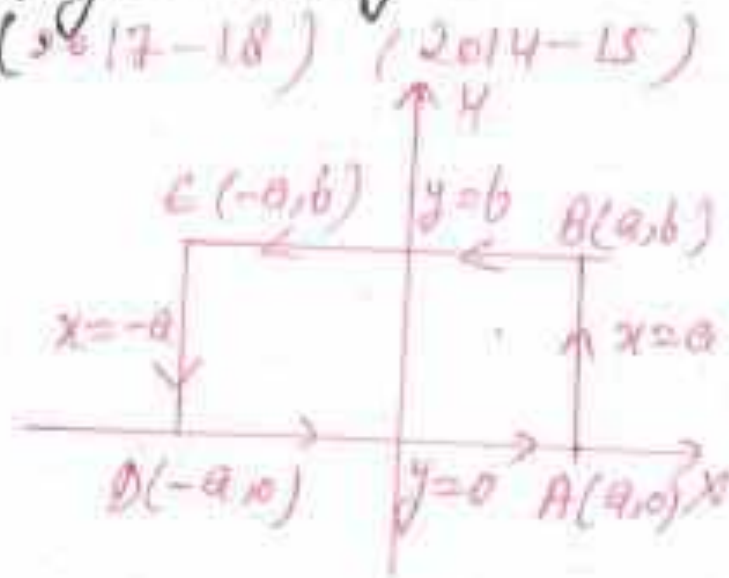
$$= 2 \int_0^1 \left(x^2 - \frac{x^2}{2} \right) \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}.$$

Q.2: Verify Stokes theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$.

Sol: Let C denote the boundary of

the rectangle $ABED$, then

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \oint_C [(x^2 + y^2)\hat{i} - 2xy\hat{j}] (\hat{i} dx + \hat{j} dy) \\ &= \oint_C [(x^2 + y^2) dx - 2xy dy]\end{aligned}$$



The curve C consists of four lines AB , BE , ED and DA .

Along AB , $x = a$, $dx = 0$ and y varies from 0 to b .

$$\therefore \int_{AB} [(x^2 + y^2) dx - 2xy dy] = \int_0^b -2ay dy = -a[y^2]_0^b = -ab^2 \quad \text{--- (1)}$$

Along BE, $y=b$, $dy=0$ and x varies from a to $-a$.

$$\begin{aligned}\therefore \int_{BE} [(x^2+y^2)dx - 2xydy] &= \int_a^{-a} (x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x \right]_a^{-a} \\ &= \frac{-2a^3}{3} - 2ab^2 \quad \text{--- (2)}\end{aligned}$$

Along ED, $x=-a$, $dx=0$ and y varies from b to 0 .

$$\therefore \int_{ED} [(x^2+y^2)dx - 2xydy] = \int_b^0 2aydy = a[y^2]_b^0 = -ab^2 \quad \text{--- (3)}$$

Along DA, $y=0$, $dy=0$ and x varies from $-a$ to a .

$$\therefore \int_{DA} [(x^2+y^2)dx - 2xdxdy] = \int_{-a}^a x^2dx = \frac{2a^3}{3} \quad \text{--- (4)}$$

Adding (1), (2), (3) and (4), we get

$$\oint_C \vec{F} \cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \quad \text{--- (5)}$$

Now $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\hat{k} = -4y\hat{k}$

For the surface S , $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} \cdot \hat{n} = -4y\hat{k} \cdot \hat{k} = -4y$$

$$\therefore \iint_S \text{curl } \vec{F} \cdot \hat{n} \, dS = \int_0^b \int_{-a}^a -4y \, dx \, dy = \int_0^b -4y [x]_{-a}^a \, dy$$

$$= -8a \int_0^b y \, dy = -8a \left[\frac{y^2}{2} \right]_0^b = -4ab^2 \quad \text{--- (6)}$$

The equality of (5) and (6) verifies Stoke's theorem.

Q.3: → Verify Stokes theorem $\vec{F} = (2y+z, x-z, y-x)$ taken over the triangle ABC cut from the plane $x+y+z=1$ by the coordinate planes. (2016-17)

Solⁿ: → By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds \quad \text{--- (1)}$$

Taking LHS,

$$\oint_{ABC} \vec{F} \cdot d\vec{s} = \int_{AB} \vec{F} \cdot d\vec{s} + \int_{BC} \vec{F} \cdot d\vec{s} + \int_{CA} \vec{F} \cdot d\vec{s}$$

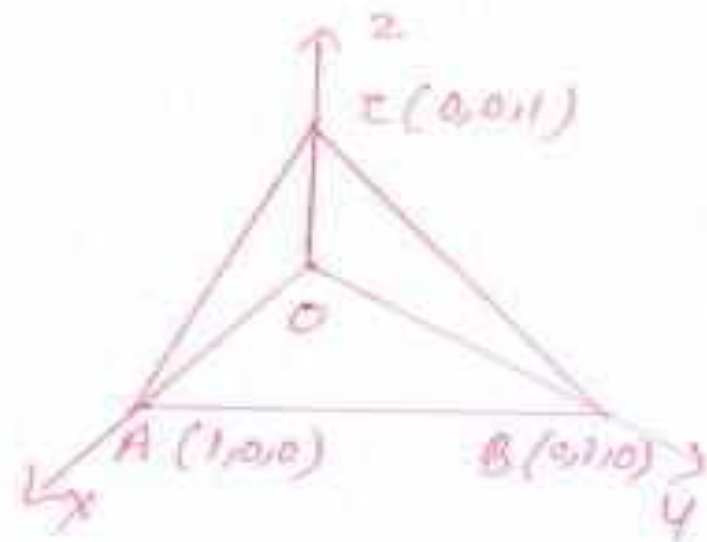
Along AB, $z=0$, $x+y=1$, $y=1-x$, $dy=-dx$

and $\vec{s} = x\hat{i} + y\hat{j}$

$$\int_{AB} \vec{F} \cdot d\vec{s} = \int_{AB} [2y\hat{i} + x\hat{j} + (y-x)\hat{k}] (\hat{i}dx + \hat{j}dy)$$

$$= \int_{AB} 2y dx + x dy = \int_{AB} 2(1-x) dx + x(-dx)$$

($\because y=1-x$ and $dy=-dx$)



$$= \int_{AB} 2 dx - 2x dx - x dx = \int_1^0 (2 - 3x) dx = \left[2x - \frac{3x^2}{2} \right]_1^0 = -\frac{1}{2}$$

Along BC, $x=0$, $y+z=1$, $z=1-y$, $dz = -dy$ and $\vec{a} = y\hat{j} + z\hat{k}$

$$\int_{BC} \vec{F} \cdot d\vec{a} = \int_{BC} [(2y+z)\hat{i} - z\hat{j} + y\hat{k}] (y\hat{j} + z\hat{k})$$

$$= \int_{BC} -z dy + y dz = \int_{BC} -(1-y) dy + y(-dy) = \int_{BC} -dy + y dy - y dy$$

$$= \int_1^0 -dy = [-y]_1^0 = 0 - (-1) = 1$$

Along CA, $y=0$, $x+z=1 \Rightarrow x=1-z$, $dx = -dz$ and $\vec{a} = x\hat{i} + z\hat{k}$

$$\int_{CA} \vec{F} \cdot d\vec{s} = \int_{CA} [z\hat{i} + (x-z)\hat{j} - x\hat{k}] (\hat{i}dx + \hat{k}dz) = \int_{CA} zdx - xdz$$

$$= \int_{CA} z(-dz) - (1-z)dz = \int_{Ac} -zdz - dz + zdz = \int_{Ac} -dz$$

$$= \int_1^0 -dz = -[z]_1^0 = -[0-1] = 1$$

Hence $\int_{AB} \vec{F} \cdot d\vec{s} = \frac{-1}{2} + 1 + 1 = \frac{3}{2}$

Now, curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x-z & y-x \end{vmatrix}$

$$= \hat{i} [1 - (-1)] - \hat{j} [-1 - 1] + \hat{k} [1 - 2] = 2\hat{i} + 2\hat{j} - \hat{k}$$

Equation of the plane ABC is $x+y+z=1$

Normal to the plane ABC is

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x+y+z-1) = \hat{i} + \hat{j} + \hat{k}$$

Normal unit vector, $\hat{n} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Now taking RHS of eqⁿ (1)

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = \iint_S (2\hat{i} + 2\hat{j} - \hat{k}) \cdot \left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \frac{dx \, dy}{\frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) \cdot \hat{k}} \right)$$

$$= \iint_S \frac{2+2-1}{\sqrt{3}} \frac{dx \, dy}{1/\sqrt{3}} = 3 \iint dx \, dy = 3 \times \text{area of } \triangle AOB$$

$$= 3 \times \frac{1}{2} \times 1 \times 1 = \frac{3}{2}$$

Verified

Q.4: Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ integrated around the rectangle in the plane $z=0$ and bounded by the lines $x=0, y=0, x=a, y=b$.
(2019-20)

Solⁿ: Proceed as Q.2.

Q.5: Verify Stoke's theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated around the square whose sides are $x=0, y=0, x=a, y=a$ in the plane $z=0$.
(2020-21)

Solⁿ: Proceed as Q.2.

Practice Questions

Q1 *Verify Stoke's Theorem for $\vec{F} = (x + y) \hat{i} + (2x - z) \hat{j} + (y + z) \hat{k}$ for the surface of a triangular lamina with vertices $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$.*

$$\text{Ans: } \iint_S \text{Curl } \vec{F} \cdot \hat{n} \, dS = \int_C \vec{F} \cdot d\vec{r} = 21.$$

Q2 *Verify Stoke's Theorem for*

$$\vec{F} = (y - z + 2) \hat{i} + (yz + 4) \hat{j} - (xz) \hat{k}$$

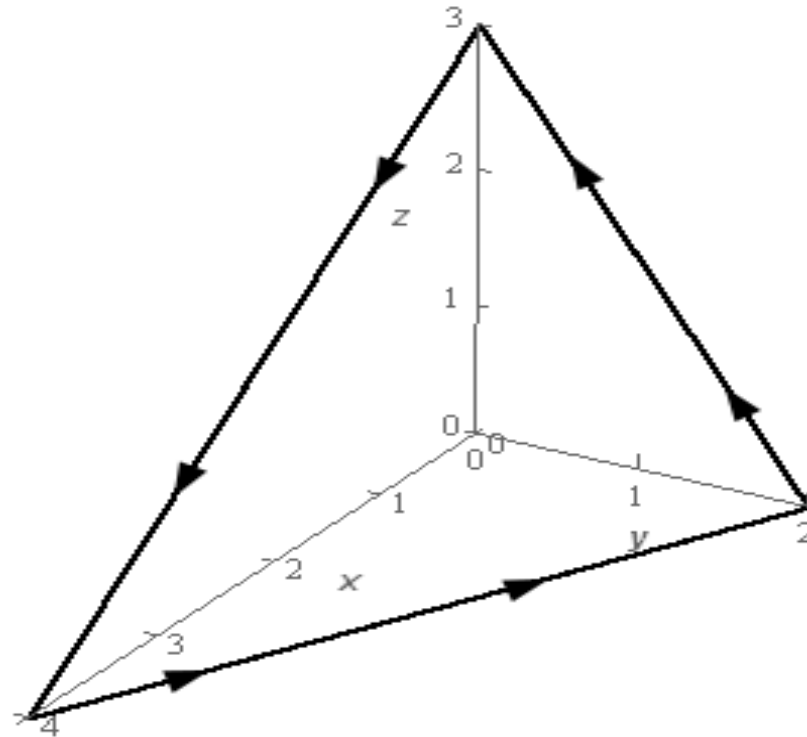
over the surface of a cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the XOY plane (open the bottom).

Ans: - 4

Q3. Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (3yx^2 + z^3) \vec{i} + y^2 \vec{j} + 4yx^2 \vec{k}$ and C

is triangle with vertices $(0,0,3)$, $(0,2,0)$ and $(4,0,0)$. C has a counter clockwise rotation if you are above the triangle and looking down towards the xy -plane. See the figure below for a sketch of the curve.

Ans -5



Q 4 Verify Stoke's theorem $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and Surface S is the portion of the sphere for $x^2 + y^2 + z^2 = 1$ above the xy -Plane.

Ans: $-\pi$