Unit- 5

Vector Calculus

Lecture 37(I)

Gradient

Scalar Point Function

If to each point P (x, y, z) of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function.

For example, the temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

Mathematically

$f(x,y,z) = x^2+2yz^5$ **is an example of scalar function**

Vector Point Function

If to each point $P(x, y, z)$ of a region R in space there corresponds a unique vector $f(P)$, then f is called a *vector point function*. The velocity of a moving fluid, gravitational force are the examples of vector point function.

Mathematically

$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$, **be an example of vector point function**

Vector Differential Operator

The vector differential operator Del is denoted by ∇. It is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

GRADIENT OF SCALAR FIELD

If $\phi\,(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is called the gradient of the scalar function ϕ.

And is denoted by grad ϕ.

Thus,

$$\text{grad}\ \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\text{gard}\ \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

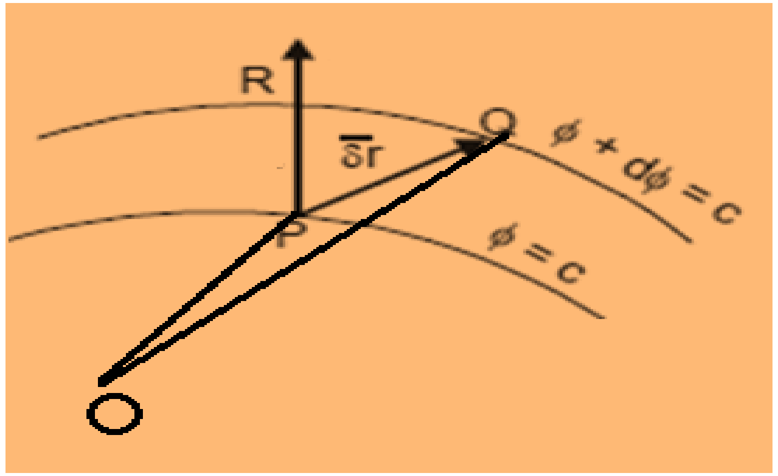
$$\text{gard}\ \phi = \nabla\ \phi$$

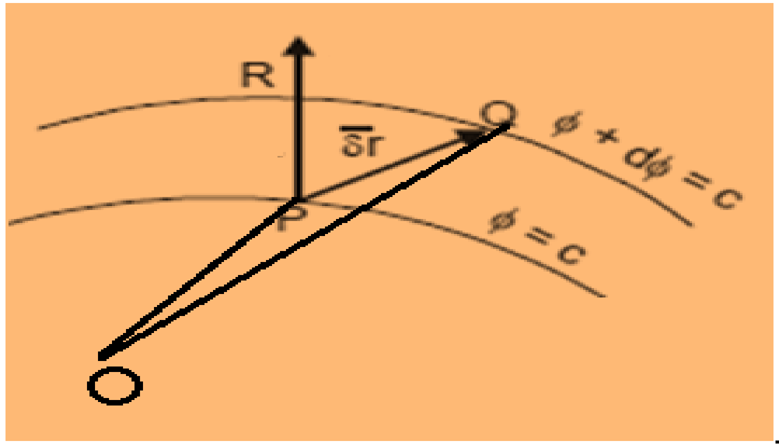
GEOMETRICAL INTERPRETATION

If a surface $\phi(x, y, z) = c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a *level surface* through P. *For example*, If $\phi(x, y, z)$ represents potential at the point P, then *equipotential surface* $\phi(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through Q, where the value of the function is $\phi + d\phi$.





Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\bar{r}$

$$\nabla\phi . d\bar{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) . (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \qquad ...(1)$$

If Q lies on the level surface of P, then $d\phi = 0$

Equation (1) becomes $\nabla\phi \,.\, dr = 0$. Then $\overline{\nabla}\phi$ is $\perp$ to $d\bar{r}$ (tangent).

Hence, $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$

PROPERTIES OF GRADIENT

(*a*) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(*b*) If ϕ_1 and ϕ_2 are two scalar point functions, then

(*i*) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(*ii*) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constant

(*iii*) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(*iv*) $\nabla\left(\dfrac{\phi_1}{\phi_2}\right) = \dfrac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$, $\phi_2 \neq 0$.

EXAMPLE 1:

Find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, – 2, – 1).

SOLUTION:

$$\text{Grad } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}\,(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k} \text{ at the point } (1, -2, -1).$$

EXAMPLE 2.

Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point

(1, 2, − 1).

SOLUTION

Let $\phi = x^3 + y^3 + 3xyz - 3$, then $\frac{\partial \phi}{\partial x} = 3x^2 + 3yz$, $\frac{\partial \phi}{\partial y} = 3y^2 + 3xz$, $\frac{\partial \phi}{\partial z} = 3xy$

$$\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

At (1, 2, - 1), $\nabla\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$

which is a vector normal to the given surface at (1, 2, – 1).

Hence a unit vector normal to the given surface at (1, 2, – 1)

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{[(-3)^2 + (9)^2 + (6)^2]}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

EXAMPLE 3

If $\nabla\phi = (y^2 - 2xyz^3)\, i + (3 + 2xy - x^2z^3)\, j + (6z^3 - 3x^2yz^2)\, k$, *find* ϕ.

SOLUTION

Let $\vec{F} = \nabla\phi$

$$\Rightarrow \quad \vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r}$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{dy}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$\therefore \qquad d\phi = \vec{F} \cdot d\vec{r}$$

$$= \{(y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\,\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}\} \cdot (dx\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= (y^2 - 2xyz^3)\,dx + (3 + 2xy - x^2z^3)\,dy + (6z^3 - 3x^2yz^2)\,dz$$

$$= (y^2\,dx + 2xy\,dy) - (2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz) + 3\,dy + 6z^3\,dz$$

$$= d(xy^2) - \mathrm{d}(x^2\,yz^3) + d(3y) + d\left(\frac{3}{2}z^4\right)$$

$$\therefore \qquad \phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + c$$

EXAMPLE 4

If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, *prove that*

$$(\text{grad } u) \cdot [(\text{grad } v) \times (\text{grad } w)] = 0.$$

SOLUTION

We have $\text{grad } u = \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} + \frac{\partial u}{\partial z} \mathbf{k}$

$= 1 \ \mathbf{i} + 1 \ \mathbf{j} + 1 \ \mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k},$

$\text{grad } v = \frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j} + \frac{\partial v}{\partial z} \mathbf{k} = 2x \ \mathbf{i} + 2y \ \mathbf{j} + 2z \ \mathbf{k}$

and $\text{grad } w = \frac{\partial w}{\partial x} \mathbf{i} + \frac{\partial w}{\partial y} \mathbf{j} + \frac{\partial w}{\partial z} \mathbf{k}$

$\therefore$ grad $u \cdot$[(grad v)$\times$(grad w)]=scalar triple product of the vectors grad u, grad v and grad w

$$=\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$=2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix}, \text{ by } R_2+R_3$$

$$=2\,(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$=2\,(x+y+z).0 = 0$$

EXAMPLE 5

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that*

(i) $grad\ r = \dfrac{\vec{r}}{r}$

(ii) $grad\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

(iii) $\nabla r^n = nr^{n-2}\ \vec{r}$

SOLUTION

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, \quad \text{or} \quad r^2 = x^2 + y^2 + z^2$$

Differentiating partially w.r.t. x, we have $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(i)

$$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = -\frac{\vec{r}}{r}.$$

(ii)

$$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}.$$

(iii)

$$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n = \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(nr^{n-1}\cdot\frac{x}{r}\right) + \hat{j}\left(nr^{n-1}\cdot\frac{y}{r}\right) + \hat{k}\left(nr^{n-1}\cdot\frac{z}{r}\right) = nr^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = nr^{n-2}\,\vec{r}\,.$$

EXAMPLE 6

Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, – 1, 2).

SOLUTION

Let $\phi_1 = x^2 + y^2 + z^2 = 9$ and $\phi_2 = x^2 + y^2 - z = 3$

Then grad $\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and grad $\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n_1}$ = grad ϕ_1 at the point (2, – 1, 2) and $\vec{n_2}$ = grad ϕ_2 at the point (2, – 1, 2). Then

$$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \text{and} \quad \vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$$

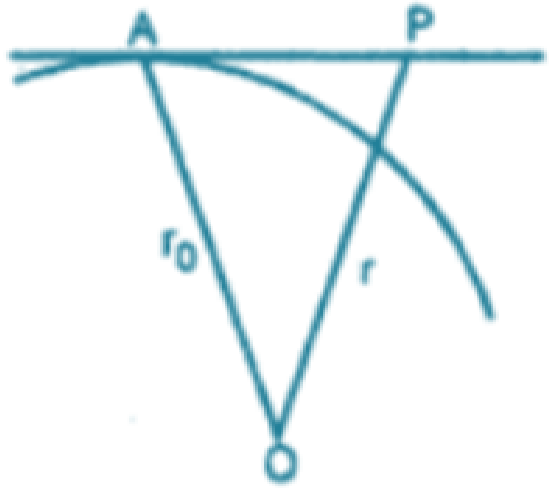
The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the two surfaces at the point (2, – 1, 2). If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16+4+16} \cdot \sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

APPLICATION OF GRADIENT
EQUATION OF THE TANGENT PLANE

Tangent Plane. Let $\mathbf{r}_0$, be the position vector of the point of contact A and $\mathbf{r}$ be the position vector of any point P on the tangent plane.



Then $\mathbf{r} - \mathbf{r}_0$ is a vector parallel to the tangent plane and grad f is normal to the tangent plane. These two are perpendicular

$$\therefore \quad (\mathbf{r} - \mathbf{r}_0) \cdot \text{grad} f = 0 \qquad (1)$$

This equation will be satisfied by any point **r** lying in the tangent plane. Moreover, for any point with position vector **r** which satisfies (1) the vector ($\mathbf{r} - \mathbf{r}_0$) is parallel to the tangent plane. It follows that $\mathbf{r} - \mathbf{r}_0$ lies in the plane; hence the end point of **r** in the plane. Therefore (1) is the equation of the tangent plane.

EXAMPLE 7

Find the equation of the tangent plane and normal to the surface $xyz = 3$ at the point (1, 2, 2).

SOLUTION

$$\text{grad} f = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k} = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Also $\mathbf{r} - \mathbf{r}_0 = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

$$= (x-1)\,\mathbf{i} + (y-2)\,\mathbf{j} + (z-2)\,\mathbf{k}.$$

The equation to the tangent plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \text{grad} f = 0$$

$\therefore$ $\{(x-1)\,\mathbf{i} + (y-2)\,\mathbf{j} + (z-2)\,\mathbf{k}\} \cdot (4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 0$

i.e., $4(x-1) + 2(y-2) + 2(z-2) = 0$

i.e., $2x + y + z = 6.$

Gradient in Polar Co-ordinate

If f(r) is a scalar function of scalar r then it's gradient is given by

$$\text{grad } f = \frac{\partial}{\partial r}(f)\cdot \hat{r}$$

$r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ex:-1 Prove that $\text{grad } r^n = n r^{n-2}\vec{r}$ where $r = |\vec{r}|$

Solⁿ:- $\text{grad } r^n = \frac{\partial}{\partial r}(r^n)\cdot\hat{r} = n r^{n-1}\hat{r} = n r^{n-2}\vec{r}$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$.

Practice Questions

Q1

If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) $\nabla f(r) = f'(r)\,\nabla r$ (ii) $\nabla \log r = \dfrac{\vec{r}}{r^2}$

Q2

If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$, show that

$$\cos\theta = \frac{3}{7\sqrt{6}}.$$

Practice Questions

Q3 Evaluate grad ϕ if $\phi = \log(x^2 + y^2 + z^2)$ **Ans.** $\dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$

Q4 Find a unit normal vector to the surface $z^2 = x^2 + y^2$ at the point (1, 0, –1). **Ans.** $\dfrac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

Lecture 37(II)

Directional Derivatives

Directional Derivatives

Directional Derivative

Let *f(x, y, z)* be a scalar valued function , directional derivative of *f*(x, y, z) at the point $\vec{a}$ in the direction of a vector $\vec{b}$ is given by

Directional derivative = $(\nabla f)_{at\,\vec{b}} \cdot \hat{a}$

$\hat{a}$ is the unit vector of the vector $\vec{a}$.

Example 1

What is the directional derivative of the function $xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$?

Solution

$$\phi(x, y, z) = xy^2 + yz^3$$

Gradient of ϕ $$= \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\, y^2 + \hat{j}\,(2xy + z^3) + \hat{k}\,(3yz^2)$$

$$\nabla\phi \text{ at } (2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$$

If $\hat{n}$ is a unit vector in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$, then $\hat{n} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \dfrac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

$\therefore$ Directional derivative of the given function ϕ at (2, −1, 1) in the direction of

$$\hat{i} + 2\hat{j} + 2\hat{k} = [\nabla\phi \text{ at } (2, -1, 1)] \cdot \hat{n}$$

$$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{1-6-6}{3} = -\frac{11}{3}$$

Example 2

Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ *at* $(1, -2, 1)$ *in the direction of* $2\hat{i} - \hat{j} - 2\hat{k}$. *Find the greatest rate of increase of* ϕ.

Solution

Here, $\phi(x, y, z) = x^2yz + 4xz^2$

Now,
$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$
$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\phi \text{ at } (1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\hat{i} + (1 \times 1)\hat{j} + \{1(-2) + 8(1)(1)\}\hat{k}$$
$$= (-4 + 4)\hat{i} + \hat{j} + (-2 + 8)\hat{k} = \hat{j} + 6\hat{k}$$

Let
$$\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

Let $\hat{a}$ = unit vector = $\dfrac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}}=\dfrac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$

So, the required directional derivative at (1, –2, 1)

$$= \nabla\phi.\hat{a} = (\hat{j}+6\hat{k}).\frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k}) = \frac{1}{3}(-1-12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = \left|\hat{j}+6\hat{k}\right| = \sqrt{1+36}$

$$= \sqrt{37}$$

Example 3

Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point (5, 0, 4).

In what direction will it be maximum ? Find also the magnitude of this maximum.

Solution

We have $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} = 2\hat{i} - 4\hat{j} + 12\hat{k}$ at P(1, 2, 3)

Also $\qquad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$

If $\hat{n}$ is a unit vector in the direction $\overrightarrow{PQ}$, then $\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

$\therefore$ Directional derivative of f in the direction $\overrightarrow{PQ} = (\nabla f) \cdot \hat{n}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{21}}[2(4) - 4(-2) + 12(1)]$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

The maximum value of this directional derivative $= |\nabla f|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}.$$

Example 4

Find the directional derivative of $\phi = e^{2x}\cos yz$ *at the origin in the direction of the tangent to the curve* $x = a\sin t,\ y = a\cos t,\ z = at$ *at* $t = \frac{\pi}{4}$.

Solution

Gradient of $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(e^{2x}\cos yz\right)$

$$= \hat{i}\left(2e^{2x}\cos yz\right) + \hat{j}\left(-e^{2x}z\sin yz\right) + \hat{k}\left\{e^{2x}\left(-\sin yz\right)y\right\}$$

At the origin i.e., when $x = 0,\ y = 0,\ z = 0.$

$$\nabla\phi = \hat{i}(2) = 2\hat{i}$$

Equation of the curve is $x = a \sin t,\ y = a \cos t,\ z = at$

Any point on the curve is $\vec{r} = \hat{i}\,(a \sin t) + \hat{j}\,(a \cos t) + \hat{k}\,(at)$

Direction of the tangent is given by $= \dfrac{d\vec{r}}{dt} = (a \cos t)\,\hat{i} - (a \sin t)\,\hat{j} + a\,\hat{k}$

At $t = \dfrac{\pi}{4}$, direction of tangent $= \dfrac{a}{\sqrt{2}}\hat{i} - \dfrac{a}{\sqrt{2}}\hat{j} + a\,\hat{k}$

$\hat{n}$ = unit direction of the tangent

$$= \frac{\dfrac{a}{\sqrt{2}}\hat{i} - \dfrac{a}{\sqrt{2}}\hat{j} + a\,\hat{k}}{\sqrt{\dfrac{a^2}{2} + \dfrac{a^2}{2} + a^2}} = \frac{\dfrac{a}{\sqrt{2}}\left(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}\right)}{\sqrt{2}\,a} = \frac{1}{2}\left(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}\right)$$

Directional derivative of ϕ at (0, 0, 0) in the direction of tangent at $t = \frac{\pi}{4}$ is $= \nabla\phi \bullet \hat{n}$ at (0, 0, 0).

$$= 2\hat{i} \bullet \frac{1}{2}\left(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}\right) = 1 .$$

Example 5

Find the directional derivative of $\nabla\cdot(\nabla f)$ *at the point (1, – 2, 1) in the direction of the normal to the surface* $xy^2z = 3x + z^2$, *where* $f = 2x^3y^2z^4$.

Solution

Here, we have

$$f = 2x^3y^2z^4$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4) = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\nabla\cdot(\nabla f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Gradient of $\nabla\cdot(\nabla f)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz4 + 48x3yz2)\hat{j}$$

$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

Gradient of $\nabla\cdot(\nabla f)$ at $(1, -2, 1) = (48 + 12 + 288)\hat{i} + (-48 - 96)\hat{j} + (192 + 16 + 192)\hat{k}$

$$= 348\hat{i} - 144\hat{j} + 400\hat{k}$$

Normal to $(xy^2z - 3x - z^2) = \nabla(xy^2z - 3x - z^2)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

Normal at $(1, -2, 1) = \hat{i} - 4\hat{j} + 2\hat{k}$

Unit Normal Vector $= \dfrac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1 + 16 + 4}} = \dfrac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$

Directional derivative in the direction of normal

$$= (348\hat{i} - 144\hat{j} + 400\hat{k}) \cdot \frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{21}}(348 + 576 + 800) = \frac{1724}{\sqrt{21}}$$

Q.7:→ Find the directional derivative of v^2, where $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point (2,0,3) in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at the point (3,2,1).

(2012-13)

Sol^n:→ $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ then $v^2 = x^2y^4 + z^2y^4 + x^2z^4 \equiv f$

Now

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= \hat{i}(2xy^4 + 2xz^4) + \hat{j}(4x^2y^3 + 4z^2y^3) + \hat{k}(2zy^4 + 4x^2z^3)$$

At point (2,0,3), $\nabla f = 324\hat{i} + 432\hat{k}$

Normal to the sphere $x^2+y^2+z^2=14 \equiv \phi$ is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At (3,2,1) $\nabla\phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Normal to the sphere $x^2+y^2+z^2=14 \equiv \phi$ is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At $(3,2,1)$ $\nabla\phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

If $\hat{n}$ is a unit vector in outward normal to the sphere then

$$\hat{n} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \frac{1}{\sqrt{56}}(6\hat{i}+4\hat{j}+2\hat{k})$$

$\therefore$ Directional derivative of f in the outward normal to the sphere $= \nabla f \cdot \hat{n}$

$$= (324\hat{i} + 432\hat{k})\frac{1}{\sqrt{56}}(6\hat{i}+4\hat{j}+2\hat{k}) = \frac{1944+864}{\sqrt{56}} = \frac{1404}{\sqrt{14}}$$

Q.2:→ Find the directional derivative of $\phi = (x^2+y^2+z^2)^{-1/2}$ at the point $(3,1,2)$ in the direction of the vector $yz\hat{i}+zx\hat{j}+xy\hat{k}$.

(2013-14)

Sol^n:→ $\phi = (x^2+y^2+z^2)^{-1/2}$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{i}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\right]+\hat{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)\right\}$$

$$+\hat{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\right\}$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} = -\frac{3\hat{i}+\hat{j}+2\hat{k}}{14\sqrt{14}} \quad \text{at } (3,1,2)$$

Let $\hat{a}$ be the unit vector in the given direction, then

$$\hat{a} = \frac{yz\hat{i}+zx\hat{j}+xy\hat{k}}{\sqrt{y^2z^2+z^2x^2+x^2y^2}} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7} \quad \text{at } (3,1,2)$$

$\therefore$ Directional derivative $= \hat{a} \cdot \nabla\phi$

$$= \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7}\left(-\frac{3\hat{i}+\hat{j}+2\hat{k}}{14\sqrt{14}}\right)$$

$$= -\frac{6+6+6}{7\cdot 14\sqrt{14}} = -\frac{9}{49\sqrt{14}}.$$

Q.3:→ Find the directional derivative of $\left(\frac{1}{r^2}\right)$ in the direction of $\vec{r}$, where $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$.

(2016-17)

Solⁿ:→ $\nabla\left(\frac{1}{r^2}\right) = \frac{2}{r^3}\hat{r} = -\frac{2}{r^4}\cdot\vec{r}$ $\quad\left(\because \hat{r} = \frac{\vec{r}}{|r|}\right)$

Let $\hat{e}$ be the unit vector in the direction of $\vec{r}$ then $\hat{e} = \hat{r} = \frac{\vec{r}}{r}$

$\therefore$ Directional derivative $= \nabla\left(\frac{1}{r^2}\right)\cdot\hat{e} = -\frac{2}{r^4}\cdot\vec{r}\cdot\frac{\vec{r}}{r} = -\frac{2}{r^3}$

Q.4:→ Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point P(1,1,1) in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}.$$

(2018-19)

Solⁿ:→

$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$$\therefore \text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(5x^2y - 5y^2z + \frac{5}{2}z^2x\right)$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \quad \text{at } (1,1,1)$$

Here $\hat{a} = \dfrac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$

$\therefore$ Directional derivative $= (\text{grad}\,\phi)\cdot\hat{a}$

$$= \left(\frac{25}{2}\hat{i} - 5\hat{j}\right)\cdot\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$$

$$= \frac{25}{3} + \frac{10}{3} = \frac{35}{3}.$$

Q.5:→ Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$. Find also the greatest rate of increase of ϕ.

(2019-20)

Solⁿ:→

$$\phi(x,y,z) = x^2yz + 4xz^2$$

$$\therefore \quad \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

At $(1,-2,1)$, $\nabla\phi = \hat{j} + 6\hat{k}$

If $\hat{n}$ is a unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$, then

$$\hat{n} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$$

So the required directional derivative at $(1,-2,1)$

$$= \nabla\phi\cdot\hat{n} = (\hat{j}+6\hat{k})\cdot\frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k}) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = |\hat{j}+6\hat{k}| = \sqrt{1+36} = \sqrt{37}$

Q.7:→ Find the directional derivative of v^2, where $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point (2,0,3) in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at the point (3,2,1).

(2012-13)

Solⁿ:→ $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ then $v^2 = x^2y^4 + z^2y^4 + x^2z^4 \equiv f$

Now

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= \hat{i}(2xy^4 + 2xz^4) + \hat{j}(4x^2y^3 + 4z^2y^3) + \hat{k}(2zy^4 + 4x^2z^3)$$

At point (2,0,3), $\nabla f = 324\hat{i} + 432\hat{k}$

Normal to the sphere $x^2+y^2+z^2=14 \equiv \phi$ is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At (3,2,1) $\nabla\phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Normal to the sphere $x^2+y^2+z^2=14 \equiv \phi$ is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14) = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

At $(3,2,1)$ $\nabla\phi = 6\hat{i}+4\hat{j}+2\hat{k}$

If $\hat{n}$ is a unit vector in outward normal to the sphere

then $\hat{n} = \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \dfrac{1}{\sqrt{56}}(6\hat{i}+4\hat{j}+2\hat{k})$

$\therefore$ Directional derivative of f in the outward normal to the sphere $= \nabla f\cdot\hat{n}$

$$= (324\hat{i}+432\hat{k})\frac{1}{\sqrt{56}}(6\hat{i}+4\hat{j}+2\hat{k}) = \frac{1944+864}{\sqrt{56}} = \frac{1404}{\sqrt{14}}$$

Q.2:→ Find the directional derivative of $\phi = (x^2+y^2+z^2)^{-1/2}$ at the point (3,1,2) in the direction of the vector $yz\hat{i}+zx\hat{j}+xy\hat{k}$.

(2013-14)

Solⁿ:→ $\phi = (x^2+y^2+z^2)^{-1/2}$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{i}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\right]+\hat{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)\right\}$$

$$+\hat{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\right\}$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} = -\frac{3\hat{i}+\hat{j}+2\hat{k}}{14\sqrt{14}} \text{ at } (3,1,2)$$

Let $\hat{a}$ be the unit vector in the given direction, then

$$\hat{a} = \frac{yz\hat{i}+zx\hat{j}+xy\hat{k}}{\sqrt{y^2z^2+z^2x^2+x^2y^2}} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7} \text{ at } (3,1,2)$$

$\therefore$ Directional derivative $= \hat{a} \cdot \nabla\phi$

$$= \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7}\left(-\frac{3\hat{i}+\hat{j}+2\hat{k}}{14\sqrt{14}}\right)$$

$$= -\frac{6+6+6}{7\cdot 14\sqrt{14}} = -\frac{9}{49\sqrt{14}}.$$

<u>Q.3:-</u> Find the directional derivative of $\left(\frac{1}{r^2}\right)$ in the direction of $\vec{r}$, where $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$.

(2016-17)

<u>Solⁿ:-</u> $\nabla\left(\frac{1}{r^2}\right) = \frac{2}{r^3}\hat{r} = -\frac{2}{r^4}\cdot\vec{r}$ $\quad\left(\because \hat{r} = \frac{\vec{r}}{|r|}\right)$

Let $\hat{e}$ be the unit vector in the direction of $\vec{r}$ then $\hat{e} = \hat{r} = \frac{\vec{r}}{r}$

$\therefore$ Directional derivative $= \nabla\left(\frac{1}{r^2}\right)\cdot\hat{e} = -\frac{2}{r^4}\cdot\vec{r}\cdot\frac{\vec{r}}{r} = -\frac{2}{r^3}$

Q.4:→ Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}.$$

(2018-19)

Solⁿ:→

$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$$\therefore \quad \text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(5x^2y - 5y^2z + \frac{5}{2}z^2x\right)$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \quad \text{at } (1,1,1)$$

Here

$$\hat{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$$\therefore \text{ Directional derivative} = (\text{grad}\,\phi)\cdot\hat{a}$$

$$= \left(\frac{25}{2}\hat{i} - 5\hat{j}\right)\cdot\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$$

$$= \frac{25}{3} + \frac{10}{3} = \frac{35}{3}.$$

Q.5:→ Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$. Find also the greatest rate of increase of ϕ.

(2019-20)

Solⁿ:→ $\phi(x,y,z) = x^2yz + 4xz^2$

$$\therefore \quad \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

At $(1,-2,1)$, $\nabla\phi = \hat{j} + 6\hat{k}$

If $\hat{n}$ is a unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$, then

$$\hat{n} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$$

So the required directional derivative at $(1,-2,1)$

$$= \nabla\phi\cdot\hat{n} = (\hat{j}+6\hat{k})\cdot\frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k}) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = |\hat{j}+6\hat{k}| = \sqrt{1+36} = \sqrt{37}$

Practice Questions

1. Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$.

Ans: $-3\sqrt{2}$

2. *Find the directional derivative of* $\frac{1}{r}$ *in the direction* $\overline{r}$ *where* $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans: $\frac{1}{r^2}$

Practice Questions

3. Find the directional derivative of $f(x, y, z) = xyz$ at the point $P(1, -1, -2)$ in the direction of the vector $(2\hat{i} - 2\hat{j} + 2\hat{k})$. **Ans.** $\frac{7}{3}$

4. Find the directional derivative of the scalar function of $f(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point $(3, 1, 3)$. **Ans.** $\frac{27}{\sqrt{11}}$

Lecture 38

Divergence of a Vector Point Function

Definition

The divergence of a vector point function $\vec{F}$ is denoted by *div* $\vec{F}$ and is defined as below.

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

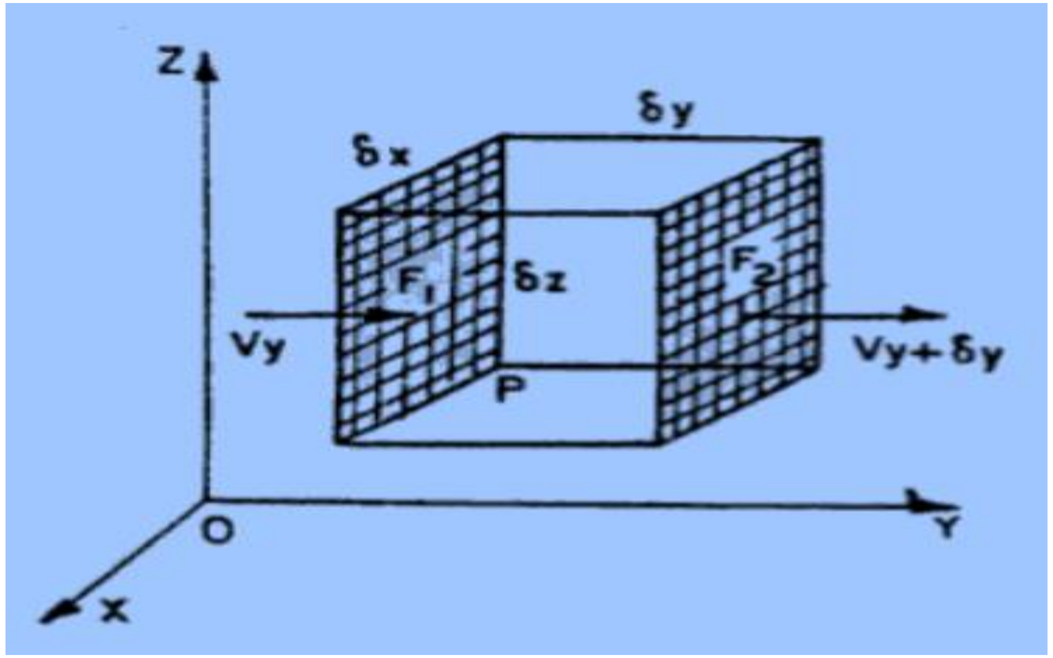
$$\operatorname{div}\vec{F} = \vec{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\operatorname{div}\vec{F}$ is a scalar function

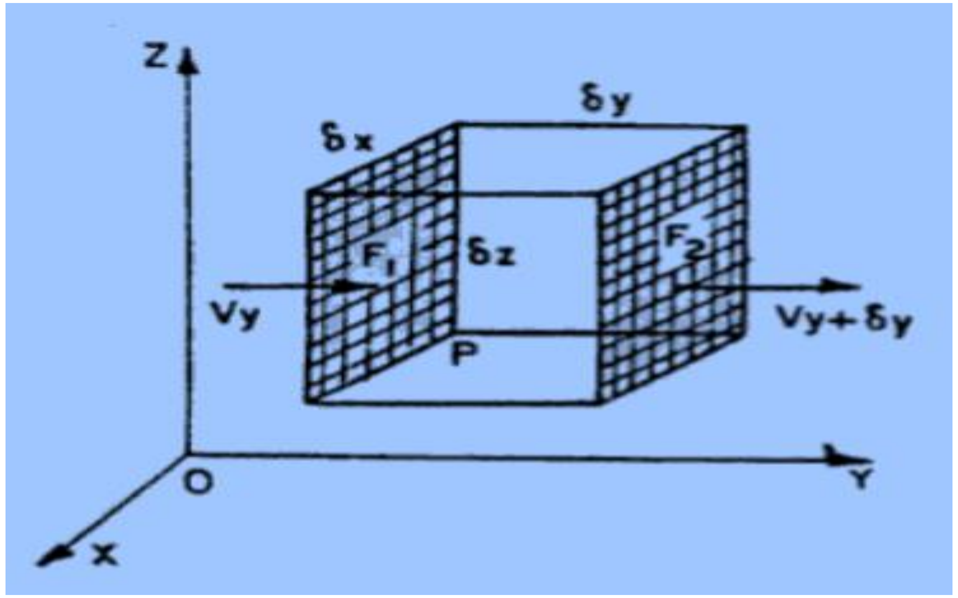
Physical interpretation of Divergence of a Vector Function

Consider a fluid having density $\rho = \rho(x, y, z, t)$ and velocity $\vec{v} = \vec{v}(x, y, z, t)$ at a point (x, y, z) at time t. Let $\vec{V} = \rho\vec{v}$, then $\vec{V}$ is a vector having the same direction as $\vec{v}$ and magnitude $\rho|\vec{v}|$. It is known as *flux*. Its direction gives the direction of the fluid flow, and its magnitude gives the mass of the fluid crossing per unit time a unit area placed perpendicular to the direction of flow.

Consider the motion of the fluid having velocity $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes with one of its corners at P.



The mass of the fluid entering through the face F_1 per unit time is $V_y\,\delta x\,\delta z$ and that flowing out through the opposite face F_2 is $V_{y+\delta y}\,\delta x\,\delta z = \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z$ by using Taylor's series.



∴ The net decrease in the mass of fluid flowing across these two faces

$$= \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z - V_y\,\delta x\,\delta z = \frac{\partial V_y}{\partial y}\delta x\,\delta y\,\delta z$$

Similarly, considering the other two pairs of faces, we get the total decrease in the mass of fluid inside the parallelopiped per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x\,\delta y\,\delta z.$

Dividing this by the volume $\delta x\ \delta y\ \delta z$ of the parallelopiped, we have the rate of loss of fluid per unit *volume*

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \operatorname{div} \vec{V}$$

Hence div $\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

Thus it can be concluded that

div $\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\vec{V} = 0$ and $\vec{V}$ is called a solenoidal vector function. Which is known in Hydrodynamics as the equation of continuity for incompressible fluids.

Note : Vectors having zero divergence are called solenoidal and are useful in various branches of physics and Engineering.

Example 1

Find the divergence of $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at the point* $(2, -1, 1)$.

Solution

$$\text{Div } \vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

Example 2

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Prove that

(i) div $\vec{r} = 3$ i.e, $\nabla . \vec{r} = 3$

(ii) div $(\vec{a} \times \vec{r}) = 0$

Solution

(i) div $\vec{r} = \nabla . \vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \qquad \because \hat{i}.\hat{i} = 1,\ \hat{i}.\hat{j} = 0 \text{ etc.}$$

$$= 1+1+1$$

$$= 3$$

(ii) $\Delta.(\vec{a} \times \vec{r}) = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\{\hat{i}\,(a_3y-a_2z) - \hat{j}\,(a_3x-a_1z) + \hat{k}\,(a_2x-a_1y)\}$

$$= -\frac{\partial}{\partial x}(a_3y-a_2z) + \frac{\partial}{\partial y}(a_3x-a_1z) - \frac{\partial}{\partial z}(a_2x-a_1y)$$

$= 0$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

Example 3

$$If\ \vec{v} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}},\ find\ the\ value\ of\ div\ \vec{v}.$$

Solution

We have, $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\text{div}\ \vec{v} = \vec{\nabla}.\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

$$= \frac{\partial}{\partial x}\frac{x}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{\left[(x^2 + y^2 + z^2)^{1/2} - x.\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x\right]}{(x^2 + y^2 + z^2)}$$

$$+\frac{\left[(x^2+y^2+z^2)^{\frac{1}{2}} - y.\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\times 2y\right]}{(x^2+y^2+z^2)} + \frac{\left[(x^2+y^2+z^2)^{1/2} - z.\frac{1}{2}(x^2+y^2+z^2)^{-1/2}.2z\right]}{(x^2+y^2+z^2)}$$

$$= \frac{(x^2+y^2+z^2)-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{(x^2+y^2+z^2)-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{(x^2+y^2+z^2)-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{(x^2+y^2+z^2)}}$$

Example 4

Prove that a vector field $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is solenoidal

Solution

A vector $\vec{F}$ is said to be solenoidal if div $\vec{F} = 0$

Here div $\vec{F} = \nabla . \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial x}\right) \cdot \{(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}\}$$

$$= \frac{\partial}{\partial x}(x^2 - y^2 + x) - \frac{\partial}{\partial y}(2xy + y)$$

$$= (2x + 1) - (2x + 1)$$

$$= 0$$

$$\Rightarrow \text{div } \vec{F} = 0$$

Thus the given vector is solenoidal

Example 5

If vector $\vec{F} = 3x\hat{i} + (x+y)\hat{j} - az\hat{k}$ is solenoidal. Find a.

Solution

A vector $\vec{F}$ is said to be solenoidal, if div $\vec{F} = 0$

$$\therefore \text{div } \vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial x}(-az)$$

$$= 3 + 1 - a = 0$$

$\therefore$ a= 4 Answer.

Example 6

If r and $\vec{r}$ have their usual meanings, show that $\operatorname{div} r^n \vec{r} = (n+3)r^n$

Solution

Since $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ so; we have

$$r^n \vec{r} = r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k}$$

$$\therefore \operatorname{div} r^n \vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n.1 + nr^{n-1}\frac{\partial r}{\partial x}x + r^n.1 + nr^{n-1}\frac{\partial r}{\partial y}y + r^n.1 + nr^{n-1}z\frac{\partial r}{\partial z}$$

$$= 3\, r^n + n r^{n-1} \left(x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right)$$

$$= 3\, r^n + n r^{n-1} \left(x \frac{x}{r} + y \frac{y}{r} + z \frac{z}{r} \right)$$

$$= 3 r^n + n r^{n-1} \left(\frac{x^2 + y^2 + z^2}{r} \right)$$

$$= 3 r^n + n r^n$$

$$= (n+3) r^n$$

Example 7

Find div $\vec{F}$ where F = grad $(x^3 + y^3 + z^3 - 3xyz)$.

Ans. div $\vec{F} = 6(x + y + z)$.

Q-6 Find the directional derivative of scalar function $f(x,y,z) = xyz$ at point $P(1,1,3)$ in the direction of the upward drawn normal to the sphere $x^2+y^2+z^2 = 11$ through the point P. (2022-23)

Soln. $f = xyz$.

Now, $\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz)$

$= yz\hat{i} + xz\hat{j} + xy\hat{k}$

at $P(1,1,3)$ $\nabla f = 3\hat{i} + 3\hat{j} + \hat{k}$

Normal to the sphere $x^2+y^2+z^2 = 11$

i.e., $\phi \equiv x^2+y^2+z^2-11$

$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2-11)$

$= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

At $P(1,1,3)$, $\nabla\phi = 2\hat{i} + 2\hat{j} + 6\hat{k}$

If $\hat{n}$ is a unit vector normal to the sphere

then $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{4+4+36}}$

$= \dfrac{2\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{44}} = \dfrac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}$

$\therefore$ Directional derivative of f in the upward normal to the sphere $= \nabla f \cdot \hat{n}$

$= (3\hat{i} + 3\hat{j} + \hat{k}) \left(\dfrac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}\right)$

$= \dfrac{3+3+3}{\sqrt{11}} = \dfrac{9}{\sqrt{11}}$

Practice Questions

1 Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

2 Find div $\vec{F}$ where F = grad $(x^3 + y^3 + z^3 - 3xyz)$.

Ans. div $\vec{F} = 6(x + y + z)$.

3 *If* $u = x^2 + y^2 + z^2$, *and* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\bar{r})$ *in terms of u.*

Ans : 5*u*

4 If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (*i*) div $\left(\frac{\vec{r}}{|\vec{r}|^3}\right) = 0$,

Q1 If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (i) div $\left(\frac{\vec{r}}{|\bar{r}|^3}\right) = 0$,

Q2 *If* $u = x^2 + y^2 + z^2$, *and* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\bar{r})$ *in terms of u.*

Q3 Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

Q4 A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

Is motion possible for incompressible fluid?

Lecture 39

Curl of a Vector Point Function & Vector Identities

Curl of a Vector Point Function

The curl of a vector point function is a vector quantity if $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$
Then

The curl (or rotation) of $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Physical Interpretation

Consider a rigid body rotating about a given axis through O with uniform angular velocity ω.

Let $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$

The linear velocity $\vec{V}$ of any point P(x, y, z) on the rigid body is given by

$\vec{V} = \vec{\omega} \times \vec{r}$

Where $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$ is the position vector of P

$$\therefore \vec{V} = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

$$\because \text{ curl } \vec{V} = \text{curl } (\vec{\omega} \times \vec{r}) = \nabla \times (\vec{\omega} \times \vec{r})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$$

$$= 2\left(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}\right)$$

$\therefore$ $\omega_1, \omega_2, \omega_3$ are constants

$$= 2\ \vec{\omega}$$

$$\therefore \vec{\omega} = \frac{1}{2} \text{ curl } \vec{V}$$

Thus the angular velocity at any points is equal to half the curl of linear velocity at that point of the body.

Note : If curl $\vec{V} = 0$, then $\vec{V}$ is said to be an irrotational vector, otherwise rotational. Also curl of a vector signifies rotation.

Example 1

Find the curl of $\vec{v} = (x\,y\,z)\,\hat{i} + (3x^2 y)\,\hat{j} + (xz^2 - y^2 z)\,\hat{k}$ *at* $(2,\ -1,\ 1)$

Solution

Here, we have

$$\vec{v} = (x\,y\,z)\,\hat{i} + (3x^2 y)\,\hat{j} + (xz^2 - y^2 z)\,\hat{k}$$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & 3x^2 y & xz^2 - y^2 z \end{vmatrix} = -2yz\,\hat{i} - (z^2 - xy)\,\hat{j} + (6xy - xz)\,\hat{k}$$

$$= -2yz\,\hat{i} + (xy - z^2)\,\hat{j} + (6xy - xz)\,\hat{k}$$

Curl at $(2,\ -1,\ 1)$

$$= -2(-1)(1)\,\hat{i} + \{(2)\,(-1) - 1\}\,\hat{j} + \{6(2)(-1) - 2(1)\}\,\hat{k} = 2\,\hat{i} - 3\,\hat{j} - 14\,\hat{k}$$

Example 2

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Prove that

(i) Curl $\vec{r} = \vec{0}$ i.e, $\nabla \times \vec{r} = \vec{0}$

(ii) Curl $(\vec{r} \times \vec{a}) = -2\vec{a}$ i.e, $\nabla \times (\vec{r} \times \vec{a}) = -2\vec{a}$

Solution

(i) Curl $\vec{r} = \nabla \times \vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) \quad = \vec{0}$$

(ii) Let us suppose that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore \vec{r} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= \hat{i}\ (a_3y - a_2z) - \hat{j}\,(a_3x - a_1z) + \hat{k}\ (a_2x - a_1y)$

Therefore, we have

$$\nabla \times (\vec{r} \times \vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$= \hat{i}\ (-a_1 - a_1) - \hat{j}\,(a_2 + a_2) + \hat{k}\ (-a_3 - a_3)$

$= -2a_1\hat{i} - 2a_2\hat{j} - 2a_3\hat{k}$

$= -2\,(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$= -2\,\vec{a}$

Example 3

A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$, show that the motion is irrotational and hence find velocity potential.

Solution

We have $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = (1-1)\hat{i} + (1-1)\hat{j} + (1-1)\hat{k} = \vec{0}$$

Hence $\vec{V}$ is irrotational

Now, if ϕ is a scalar potential then, we have

$\vec{V} = \nabla\phi$

$$\Rightarrow (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get

$$\frac{\partial \phi}{\partial x} = y+z, \quad \frac{\partial \phi}{\partial y} = z+x \;\&\; \frac{\partial \phi}{\partial z} = x+y$$

$$\text{Also } d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= (y+z)dx + (z+x)dy + (x+y)dz$$

$$= ydx + zdx + zdy + xdy + xdz + ydz$$

$$= ydx + xdy + zdy + ydz + xdz + zdx$$

$$= d(xy) + d(yz) + d(xz)$$

Interating term by term we get

$$\phi = xy + yz + xz + \text{constant}$$

Example 4

Find the constants a, b, c, so that

$$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

is irrotational.

Solution

We have,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$$

$$= (c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k}$$

As $\vec{F}$ is irrotational, $\nabla \times \vec{F} = \vec{0}$

i.e., $(c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore \quad c+1 = 0, \quad 4-a = 0 \quad \text{and} \quad b-2 = 0$

i.e., $a = 4, \quad b = 2, \quad c = -1$

Putting the values of a, b, c in (1), we get

$$\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

Vector Identities

1. $div\ (\mathbf{A}+\mathbf{B})=div\ \mathbf{A}+div\ \mathbf{B}$

2. $curl\ (\mathbf{A}+\mathbf{B})=curl\ \mathbf{A}+curl\ \mathbf{B}$

3. *If* $\mathbf{A}$ *is a differentiable* vector *function and* ϕ *is a differentiable scalar function, then*

$$div\ (\phi\mathbf{A})=(grad\ \phi)\cdot\mathbf{A}+\phi\ div\ \mathbf{A}$$

4. $curl\ (\phi\mathbf{A})=(grad\ \phi)\times\mathbf{A}+\phi\ curl\ \mathbf{A}$

5. $div\ (\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot curl\ \mathbf{A}-\mathbf{A}\cdot curl\ \mathbf{B}$

Vector Identities

6. $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) - (A.\nabla)B + A(\nabla.B)$
7. $\nabla \times (\nabla f) = 0$
8. $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$
9. $\nabla^2 f \triangleq \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Example 5

(*i*) Prove that the *vector* $f(r)\overline{r}$ is irrotational

(*ii*) Prove that the vector $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$ is irrotational

Solution

$$(i)\ \text{curl}\ \{f(r)\vec{r}\} = f(r)\ \text{curl}\ \vec{r} + \{\text{grad}\ f(r)\} \times \vec{r}$$

$$= \vec{0} + f'(r)\ \hat{r} \times \vec{r}$$

$$= \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \vec{0}$$

Hence $f(r)\ \vec{r}$ is irrotational.

(*ii*) $\text{grad } f(r) = f'(r)\hat{r} = \frac{1}{r} f'(r) \vec{r}$

$\text{div }\{\text{grad } f(r)\} = \nabla^2 f(r)$

$$= \text{div}\left\{\frac{f'(r)}{r}\vec{r}\right\} = \frac{f'(r)}{r}\text{div }\vec{r} + \text{grad}\left\{\frac{f'(r)}{r}\right\}.\vec{r}$$

$$= \frac{3}{r}f'(r) + \left\{\frac{rf''(r) - f'(r)}{r^2}\right\}\hat{r}.\vec{r}$$

$$= \frac{3}{r}f'(r) + \left\{\frac{rf''(r) - f'(r)}{r^3}\right\}(\vec{r}.\vec{r})$$

$$= \frac{3}{r}f'(r) + \left\{\frac{rf''(r) - f'(r)}{r}\right\}$$

$\Rightarrow$ $$\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$$

Now, $$\nabla^2 \log r = -\frac{1}{r^2} + \frac{2}{r}.\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{x^2 + y^2 + z^2}$$

Q. i) If $\vec{A} = (xz^2\hat{i} + 2y\hat{j} - 3xz\hat{k})$ and $\vec{B} = (3xz\hat{i} + 2yz\hat{j} - z^2\hat{k})$.

Find the value of $[\vec{A}\times(\nabla\times\vec{B})]$ & $[(\vec{A}\times\nabla)\times\vec{B}]$. (2016-17)

Sol^n: (i) $[\vec{A}\times(\nabla\times\vec{B})]$

$$\nabla\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2yz & -z^2 \end{vmatrix} = \hat{i}[0-2y] - \hat{j}[0-3x] + \hat{k}[0-0]$$

$$= -2y\hat{i} + 3x\hat{j} + 0\hat{k}$$

Now

$$[\vec{A}\times(\nabla\times\vec{B})] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ -2y & 3x & 0 \end{vmatrix} = \hat{i}[0+9x^2z] - \hat{j}[0-6xyz] + \hat{k}[3x^2z^2+4y^2]$$

$$= 9x^2z\,\hat{i} + 6xyz\,\hat{j} + (3x^2z^2+4y^2)\hat{k}$$

(ii) $[(\vec{A}\times\nabla)\times\vec{B}]$

$$\vec{A}\times\nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}[0+0]-\hat{j}[2xz+3z]+\hat{k}[0-0]$$

$$= 0\hat{i}-(2xz+3z)\hat{j}+0\hat{k}$$

$$[(\vec{A}\times\nabla)\times\vec{B}] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -(2xz+3z) & 0 \\ 3xz & 2yz & -z^2 \end{vmatrix} = [z^2(2xz+3z)-0]\hat{i}-[0-0]\hat{j}$$

$$+[0+3xz(2xz+3z)]\hat{k}$$

$$= (2xz^3+3z^3)\hat{i}+(6x^2z^2+9xz^2)\hat{k}$$

Q. :→ A fluid motion is given by

$$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}.$$

Is the motion irrotational? If so, find the velocity potential. (2020-21)

Solⁿ:→ Proceed as

Q. :→ If $\vec{F} = (\vec{a}\cdot\vec{r})\vec{r}$, where $\vec{a}$ is a constant vector, find curl $\vec{F}$ and prove that it is perpendicular to $\vec{a}$. (2011-12)

Solⁿ:→ Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$$\Rightarrow (\vec{a}\cdot\vec{r})\vec{r} = (a_1x + a_2y + a_3z)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (a_1x^2 + a_2xy + a_3xz)\hat{i} + (a_1xy + a_2y^2 + a_3yz)\hat{j}$$

$$+ (a_1xz + a_2yz + a_3z^2)\hat{k}$$

Now $\text{Curl}\,(\vec{a}\cdot\vec{r})\cdot\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1x^2+a_2xy+a_3xz & a_1xy+a_2y^2+a_3yz & a_1xz+a_2yz+a_3z^2 \end{vmatrix}$

$\text{Curl}\,\vec{F} = \hat{i}(a_2z-a_3y)-\hat{j}(a_1z-a_3x)+\hat{k}(a_1y-a_2x)$

Now to show $\text{curl}\,\vec{F}$ is perpendicular to $\vec{a}$ i.e. we have to show $\text{Curl}\,\vec{F}\cdot\vec{a}=0$

$\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\text{Curl}\,\vec{F}\cdot\vec{a} = (a_2z-a_3y)a_1-(a_1z-a_3x)a_2+(a_1y-a_2x)a_3$

$= 0$

Practice Questions

(1) Find the Curl of the following vector fields

$\vec{F} = x^2y^2\hat{\imath} + 2xy\hat{\jmath} - (y^2 - xy)\hat{k}$ at (1,2,3)

Ans: Curl $\vec{F} = (2y - x)\hat{\imath} + y\hat{\jmath} + 2y(1 - x^2)\hat{k}$

(2) Find the Curl of the following vector fields

$\vec{F} = e^{xyz}(xy^2\hat{\imath} + yz^2\hat{\jmath} + zx^2\hat{k}$

Ans: Curl $\vec{F} = -39e^6\hat{\imath} + 3e^6\hat{\jmath} + 92e^6\hat{k}$

(3) *If a vector field is given by*

$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. *Is this field irrotational ? If so, find its scalar potential.*

Ans: Yes, scalar potential is $\frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$

Practice Questions

(4)

A fluid motion is given by

$$\vec{v} = (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k}$$

is the motion irrotational? If so, find the velocity potential.

Ans: Yes, Velocity potential $= xy \sin z + \cos x + y^2 z + c.$

Lecture 40

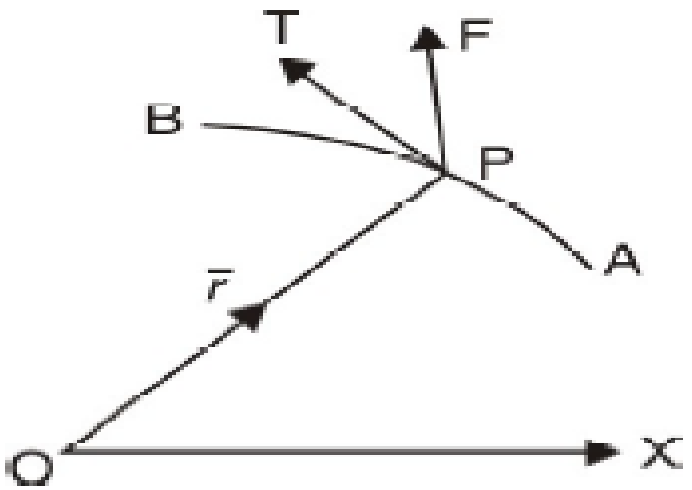
Line Integral, Surface Integral And Volume Integral

Line Integral

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along the curve AB is defined as

$$\text{Line integral} = \int_c \left(\vec{F} \cdot \frac{\vec{dr}}{ds} \right) ds = \int_c \vec{F} \cdot \vec{dr}$$



Note (1) Work. If $\overline{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F} \cdot \vec{dr}$

Example 1

If a force $\bar{F} = 2x^2 y\hat{i} + 3xy\hat{j}$ *displaces a particle in the xy-plane from (0, 0) to (1, 4) along a curve* $y = 4x^2$. *Find the work done.*

Solution

$$\text{Work done} = \int_c \vec{F} \cdot \vec{dr}$$

$$\left[\begin{array}{l} \vec{r} = x\hat{i} + y\hat{j} \\ \vec{dr} = dx\hat{i} + dy\,\hat{j} \end{array} \right]$$

$$= \int_c (2x^2 y\hat{i} + 3xy\,\hat{j}) \cdot (dx\hat{i} + dy\,\hat{j})$$

$$= \int_c (2x^2 y\,dx + 3xy\,dy)$$

Putting the values of y and dy, we get

$$\begin{pmatrix} y = 4x^2 \\ dy = 8x\,dx \end{pmatrix}$$

$$= \int_0^1 \cdot [2x^2 (4x^2)\, dx + 3x\,(4x^2)\, 8x\, dx]$$

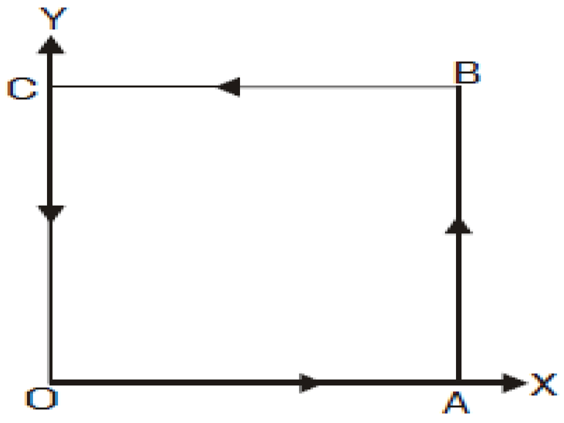
$$= 104 \int_0^1 x^4\, dx = 104 \left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$

Ans.

Example 2

Evaluate $\int_C \vec{F}\cdot\vec{dr}$ *where* $\vec{F} = x^2\hat{i} + xy\hat{j}$ *and C is the boundary of the square in the plane* $z = 0$ *and bounded by the lines* $x = 0,\ y = 0,\ x = a$ *and* $y = a$.

Solution



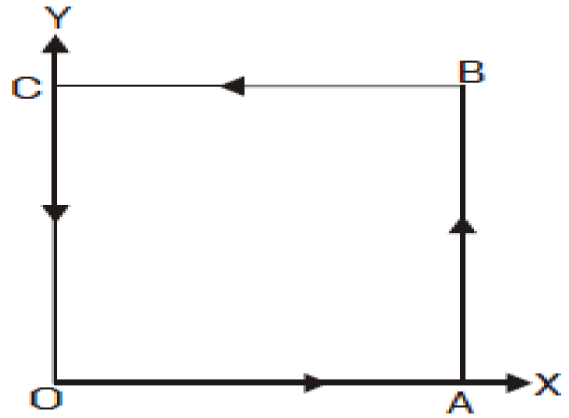
$$\int_C \vec{F}\cdot\vec{d}\,r = \int_{OA}\vec{F}\cdot\vec{dr} + \int_{AB}\vec{F}\cdot\vec{dr} + \int_{BC}\vec{F}\cdot\vec{dr} + \int_{CO}\vec{F}\cdot\vec{dr}$$

$$\vec{r} = x\hat{i} + y\hat{j},\quad \vec{d}\,r = dx\hat{i} + dy\hat{j},\quad \vec{F} = x^2\hat{i} + xy\,\hat{j}$$

$$\vec{F}\cdot\vec{dr} = x^2dx + xydy \qquad \ldots(1)$$

On $OA, y = 0$ $\therefore \vec{F}.\vec{dr} = x^2 dx$

$$\int_{OA} \vec{F}.\vec{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \quad ...(2)$$



On $AB, x = a$ $\therefore dx = 0$

(1) becomes

$$\therefore \vec{F}.\vec{dr} = aydy$$

$$\int_{Ab} \vec{F}.\vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \quad ...(3)$$

On $BC, y = a$ $\therefore dy = 0$

$\Rightarrow$ (1) becomes $\vec{F}.\vec{dr} = x^2 dx$

$$\int_{BC} \vec{F}.\vec{dr} = \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \quad ...(4)$$

On $CO, x = 0,$ $\therefore \vec{F}.\vec{dr} = 0$

(1) becomes

$$\int_{CO} \vec{F}.\vec{dr} = 0 \qquad ...(5)$$

On adding (2), (3), (4) and (5), we get $\int_C \vec{F}.\vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$

Q.1: Find the work done in moving a particle in the force field: $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve $x^2 = 4y$ and $3x^3 = 8z$ from $x = 0$ to $x = 2$.

(2011-12)

Solⁿ: Work done $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_C \left[3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}\right] \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C 3x^2 dx + (2xz - y)dy + z\,dz$$

Along the curve $x^2 = 4y$ and $3x^3 = 8z$ from $x = 0$ to $x = 2$

$\Rightarrow$ $2x\,dx = 4dy$ and $9x^2 dx = 8\,dz$

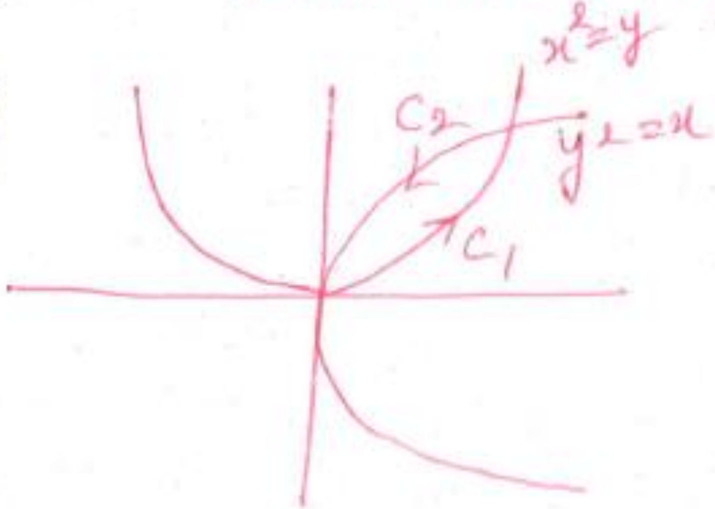
$\Rightarrow$ Work done $= \displaystyle\int_{x=0}^{2} \left[3x^2 dx + \left(2x \cdot \frac{3x^3}{8} - \frac{x^2}{4}\right)\frac{x}{2} dx + \frac{3x^3}{8} \cdot \frac{9x^2}{8} dx\right]$

$$= \int_{x=0}^{2} \left[3x^2 + \frac{1}{8}\left(3x^5 - x^3\right) + \frac{27}{64} x^5\right] dx$$

$$= \left[x^3 + \frac{1}{8}\left(\frac{3x^6}{6} - \frac{x^4}{4}\right) + \frac{27}{64} \frac{x^6}{6}\right]_0^2$$

$$= \left[8 + \frac{1}{8}(32 - 4) + \frac{27}{64} \cdot \frac{64}{6}\right] = 16$$

Q.3:→ If $\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$, evaluate $\oint_c \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y = x^2$ and $y^2 = x$. (2013-14, 2017-18)



Solⁿ:→ $\oint_c \vec{A}\cdot d\vec{r} = \oint_c (x-y)\,dx + (x+y)\,dy$

C consisting of $y = x^2$ and $y^2 = x$

$$\oint_c \vec{A}\cdot d\vec{r} = \oint_{c_1} \vec{A}\cdot d\vec{r} + \oint_{c_2} \vec{A}\cdot d\vec{r}$$

Along C_1, $y = x^2$, $dy = 2x\,dx$

$$\Rightarrow \oint_{c_1} \vec{A}\cdot d\vec{r} = \int_0^1 (x-x^2)\,dx + (x+x^2)\,2x\,dx = \int_0^1 (x+x^2+2x^3)\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^4}{4}\right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{4}{3}$$

Now Along C_2, $x = y^2$, $dx = 2y\,dy$

$$\oint_{C_2} \vec{A} \cdot d\vec{r} = \int_1^0 (y^2 - y)\,2y\,dy + (y^2 + y)\,dy$$

$$\oint_{C_2} \vec{A} \cdot d\vec{r} = \int_1^0 (2y^3 - y^2 + y)\,dy = \left[\frac{y^4}{2} - \frac{y^3}{3} + \frac{y^2}{2}\right]_1^0$$

$$= -\left[\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right] = -\frac{2}{3}$$

Thus $\oint_c \vec{A} \cdot d\vec{r} = \frac{4}{3} + \left(-\frac{2}{3}\right) = \frac{2}{3}$

Surface Integral

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

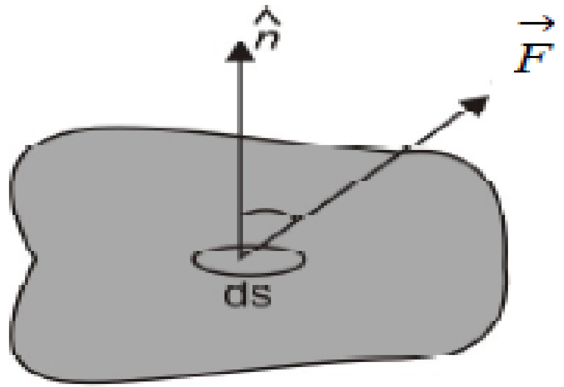
Component of $\vec{F}$ along the normal

$= \vec{F} \cdot \hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds and

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|} \qquad ds = \frac{dx\,dy}{(\hat{n} \cdot \hat{k})}$$

Surface integral of F over S

$$= \Sigma\, \vec{F} \cdot \hat{n} \qquad = \iint_S (\vec{F} \cdot \hat{n})\, ds$$



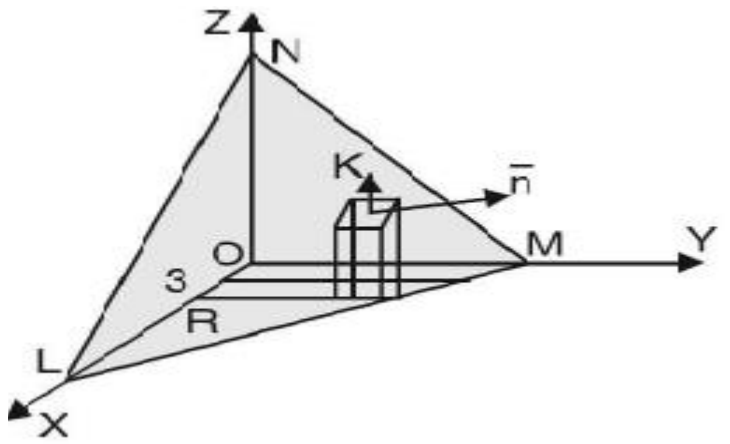
Example 3

. *Evaluate* $\iint_S \vec{A}.\hat{n}\, ds$ *where* $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.*

Solution

A vector normal to the surface "S" is given by

$$\nabla(2x + y + 2z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

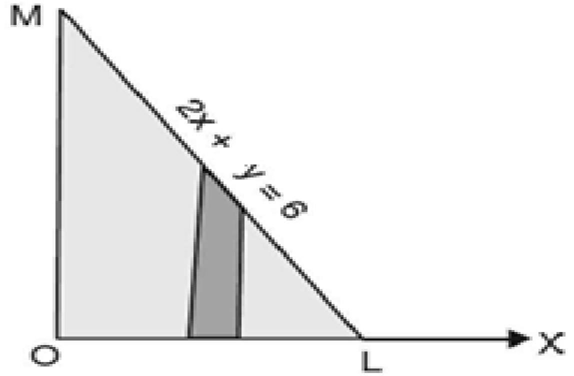


And $\hat{n}$ = a unit vector normal to surface S

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} +$$

$$\hat{k}.\hat{n} = \hat{k}.\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\therefore \qquad \iint_S \overline{A} \cdot \hat{n}\, ds = \iint_R \overline{A} \cdot \hat{n} \frac{dx\, dy}{\hat{k} \cdot \overline{n}}$$



Where R is the projection of S.

Now, $\vec{A} \cdot \hat{n} = [(x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}] \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz \qquad \text{...(1)}$$

Putting the value of z in (1), we get

$$\vec{A} \cdot \hat{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right) \qquad \left(\because \text{ on the plane } 2x + y + 2z = 6, \; z = \frac{(6 - 2x - y)}{2}\right)$$

$$\vec{A} \cdot \hat{n} = \frac{2}{3}y(y + 6 - 2x - y) = \frac{4}{3}y(3 - x) \qquad \text{...(2)}$$

Hence, $$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \bar{A}\cdot\bar{n}\,\frac{dx\,dy}{|\hat{k}\cdot\bar{n}|} \qquad ...(3)$$

Putting the value of $\vec{A}\cdot\hat{n}$ from (2) in (3), we get

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \frac{4}{3}y\,(3-x)\cdot\frac{3}{2}\,dx\,dy = \int_0^3\int_0^{6-2x} 2y\,(3-x)\,dy\,dx$$

$$= \int_0^3 2\,(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3-x)\,(6-2x)^2\,dx = 4\int_0^3 (3-x)^3\,dx$$

$$= 4.\left[\frac{(3-x)^4}{4\,(-1)}\right]_0^3 = -(0-81) = 81$$

Example 4

Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \vec{ds}$ *where S is the surface of the sphere*

$x^2 + y^2 + z^2 = a^2$ *in the first octant.*

Solution

Here, $\phi = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because\ x^2 + y^2 + z^2 = a^2]$$

miet

Here, $$\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$

$$\vec{F}\cdot\hat{n} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

Now, $$\iint_S F\cdot\hat{n}\,ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dx\,dy}{|\hat{k}\,.\,\hat{n}|} = \int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{3xyz\,dx\,dy}{a\left(\frac{z}{a}\right)}$$

$$= 3\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\,(a^2 - x^2)\,dx = \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}.$$

Volume Integral

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral = $\iiint_V \vec{F}\, dV$

Example 5

If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F}\, dv$ where, v is the region bounded by the surfaces

$$x = 0,\ y = 0,\ x = 2,\ y = 4,\ z = x^2,\ z = 2.$$

Solution

$$\iiint_V \vec{F}\, dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^4 \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dz\, dy\, dx = \int_0^2 \int_0^4 [z^2\hat{i} - xz\hat{j} + yz\,\hat{k}]_{x^2}^2\, dy\, dx$$

$$= \int_0^2 \int_0^4 [4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2 y\hat{k}]\, dy\, dx$$

$$= \int_0^2 \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - \frac{x^2 y^2}{2}\hat{k}\right]_0^4 dx$$

$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\, dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k} \right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

Practice Questions

1 Find the work done by a force $y\hat{i} + x\hat{j}$ which displaces a particle from origin to a point $(\hat{i} + \hat{j})$

Ans. 1

2 Find the work done when a force $\overline{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$ moves a particle from origin to (1, 1) along a parabola $y^2 = x$.

Ans. $\frac{2}{3}$

3 If $\vec{F} = (2x^2 - 3z)\,\hat{i} - 2xy\,\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \times \vec{F}\, dV$, where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Ans. $\frac{8}{3}(\hat{j} - \hat{k})$

4 Evaluate $\iint_S \vec{A} \cdot \hat{n}\, ds$, where $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Ans. 81

Practice Questions

5 Evaluate $\iint_S \vec{A}\,.\,\hat{n}\,ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. **Ans.** 90

6 Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where, $F = 2yx\hat{i} - yz\hat{j} + x^2\hat{k}$ over the surface S of the cube bounded by the coordinate planes and planes $x = a$, $y = a$ and $z = a$. **Ans.** $\frac{1}{2}a^4$

Thank You

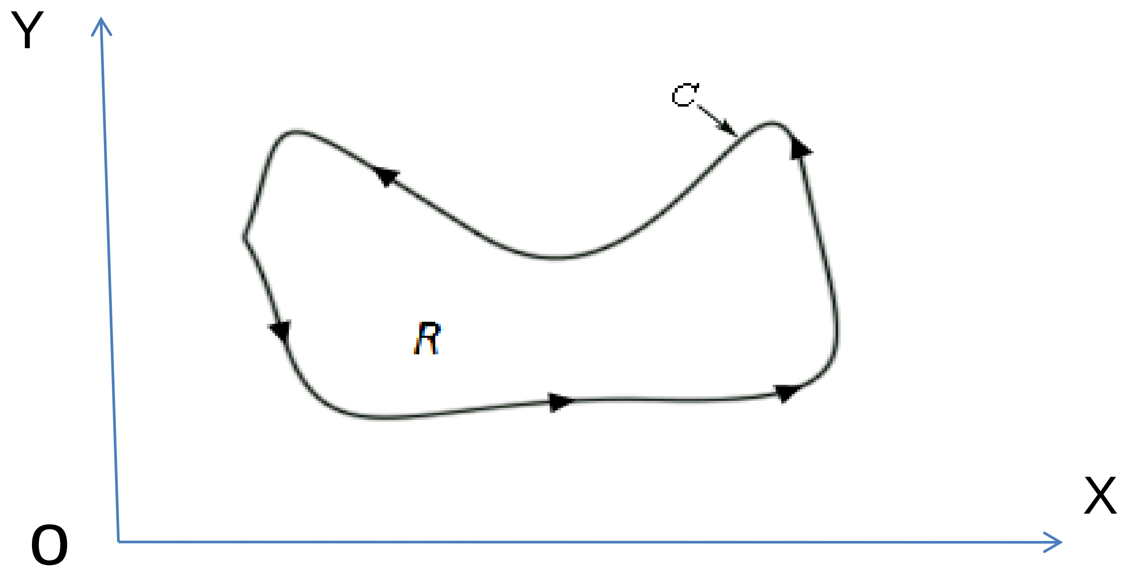
Lecture 41(I)

Green's Theorem and its Applications - I

Green Theorem

If $\phi(x, y)$, $\psi(x, y)$, $\dfrac{\partial \phi}{\partial v}$ and $\dfrac{\partial \psi}{\partial x}$ *be continuous functions over a region R bounded by simple closed curve C in x – y plane, then*

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$



Example 1

Using Green's Theorem, evaluate $\int_c (x^2\, y dx + x^2\, dy)$, *where c is the boundary described counter clockwise of the triangle with vertices (0, 0), (1, 0), (1, 1).*

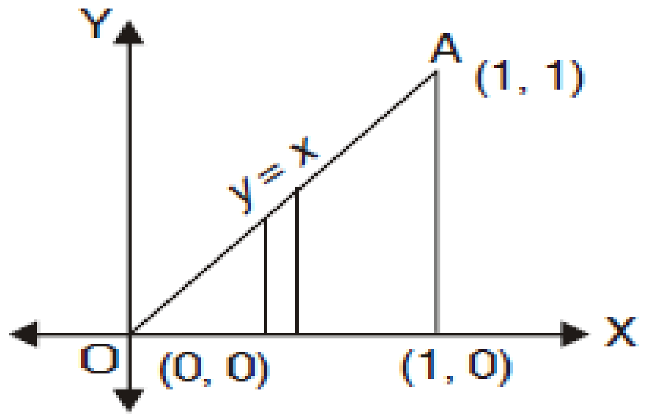
Solution

By Green's Theorem, we have

$$\int_c (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

$$\int_c (x^2 y\, dx + x^2\, dy) = \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy = \int_0^1 (2x - x^2)\, dx\, [y]_0^x$$



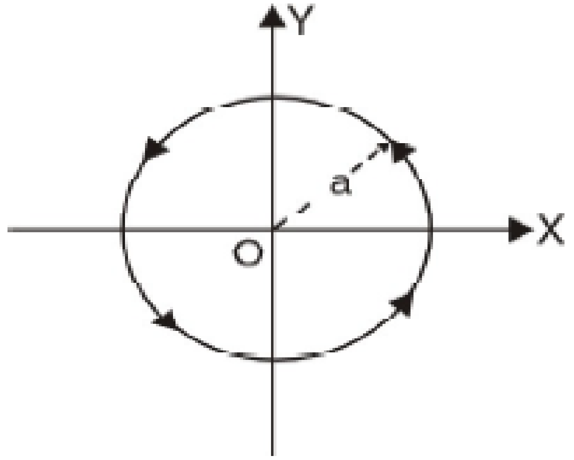
$$= \int_0^1 (2x - x^2)(x)\,dx = \int_0^1 (2x^2 - x^3)\,dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12}$$

Example 2

A vector field $\vec{F}$ *is given by* $\vec{F} = \sin y\hat{i} + x(1+\cos y)\hat{j}$.

Evaluate the line integral $\int_C \vec{F}\cdot\vec{dr}$ *where C is the circular path given by* $x^2 + y^2 = a^2$.



Solution

$$\vec{F} = \sin y\hat{i} + x(1+\cos y)\hat{j}$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_C [\sin y\hat{i} + x(1+\cos y)\hat{j}]\cdot(\hat{i}dx + \hat{j}dy) = \int_C \sin y\, dx + x(1+\cos y)\, dy$$

On applying Green's Theorem, we have

$$\oint_c (\phi\, dx + \psi\, dy) = \iint_s \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

$$= \iint_s [(1 + \cos y) - \cos y]\, dx\, dy$$

-

where s is the circular plane surface of radius a.

$$= \iint_s dx\, dy = \text{Area of circle} = \pi\, a^2.$$

Example 3

Apply Green's Theorem to evaluate $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, *where* C *is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.

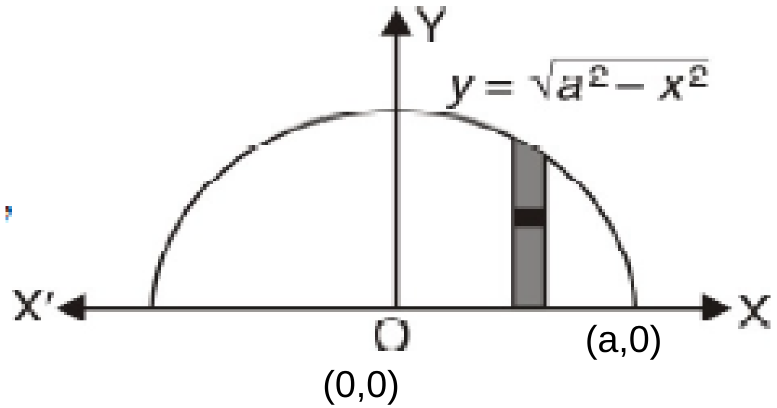
Solution

$$\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$$

By Green's Theorem, we've $\int_C (\phi\,dx + \psi\,dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

$$= \int_{-a}^{a} \int_0^{\sqrt{a^2 - x^2}} \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dx\,dy$$

$$= \int_{-a}^{a} \int_0^{\sqrt{a^2 - x^2}} (2x + 2y)\,dx\,dy = 2\int_{-a}^{a} dx \int_0^{\sqrt{a^2 - x^2}} (x + y)\,dy$$



$$= 2\int_{-a}^{a} dx \left(xy + \frac{y^2}{2} \right)_0^{\sqrt{a^2 - x^2}} = 2\int_{-a}^{a} \left(x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2} \right) dx$$

$$= 2\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx + \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$\left[\begin{array}{ll} \int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx, & f \text{ is even} \\ = 0, & f \text{ is odd} \end{array} \right]$$

$$= 0 + 2\int_{0}^{a} (a^2 - x^2)\, dx = 2\left(a^2 x - \frac{x^3}{3} \right)_0^a = 2\left(a^3 - \frac{a^3}{3} \right) = \frac{4a^3}{3}$$

Along $C_3 : y = x,\ dy = dx\ ;\ x : 1 \text{ to } 0\ ;$

$$I_3 = \int_{C_3} (xdy - ydx) = \int (xdx - xdx) = 0$$

$$A = \frac{1}{2}(I_1 + I_2 + I_3) = \frac{1}{2}(0 + 2\log 2 + 0) = \log 2$$

Example 4

$$Evaluate \oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,\ where\ C = C_1 \cup C_2\ with\ C_1 : x^2+y^2 = 1$$

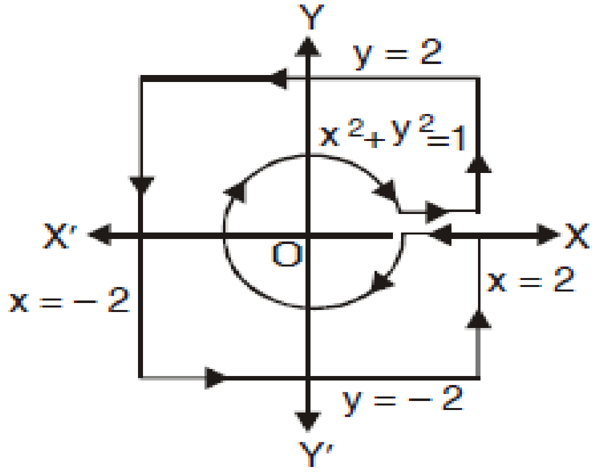
$$and\ C_2 : x = \pm 2,\ y = \pm 2.$$

Solution

$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \iint \left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \right) dx\,dy$$

$$= \iint \left[\frac{(x^2+y^2)1 - 2x\,(x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)1 - 2y\,(y)}{(x^2+y^2)^2} \right] dx\,dy$$



$$= \iint \left[\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \right] dx\,dy$$

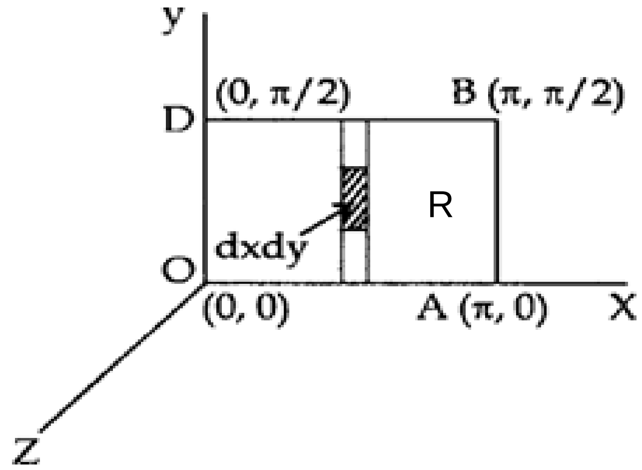
$$= \iint \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} \right] dx\,dy$$

$$= \iint \frac{0}{(x^2 + y^2)^2}\, dx\,dy = 0$$

Example 1

Evaluate by Green's theorem $\int_C [e^{-x} \sin y dx + e^{-x} \cos y dy]$ where C is the rectangle with vertices (0,0), (π,0), (π, π/2), (0, π/2) and hence verify Green's theorem.

Solution



By Green's theorem we have

$$\int_C (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Comparing the given integral

$M = e^{-x} \sin y$ and $N = e^{-x} \cos y$

Then $\frac{\partial M}{\partial y} = e^{-x} \cos y$ and $\frac{\partial N}{\partial x} = -e^{-x} \cos y$

Hence by Green's theorem

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$$\int_C [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] = \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_{x=0}^{\pi}\int_{y=0}^{\pi/2} e^{-x}\cos y\,dx\,dy$$

$$= -2\left[-e^{-x}\right]_0^{\pi}\left[\sin y\right]_0^{\pi/2}$$

$$= 2(e^{-\pi}-1)(1)$$

$$= 2(e^{-\pi}-1)$$

Evaluation of line integral:

For this purpose, let us evaluate the given line integral directly.

$$\int_C [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy]$$

$$= \int_{OA} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] + \int_{AB} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] +$$

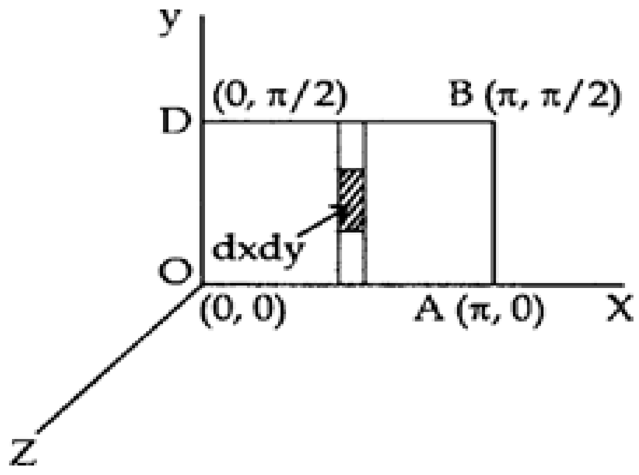
$$\int_{BD} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] + \int_{DO} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy]$$

Now along OA, $y = 0 \Rightarrow dy = 0$

along AB, $x = \pi \Rightarrow dx = 0$

along BD, $y = \pi/2 \Rightarrow dy = 0$

along DO, $x = 0 \Rightarrow dx = 0$



Hence the given line integral

$$= \quad 0 + \int_0^{\pi/2} e^{-\pi} \cos y \, dy + \int_{\pi}^{0} e^{-x} dx + \int_{\pi/2}^{0} \cos y \, dy$$

$$= \quad e^{-\pi} \left[\sin y\right]_0^{\pi/2} + \left[-e^{-x}\right]_{\pi}^{0} + \left[\sin y\right]_{\pi/2}^{0}$$

$$= \quad e^{-\pi} - (1 - e^{-\pi}) + (-1) \quad = \quad 2(e^{-\pi} - 1)$$

Hence Green's theorem is verified.

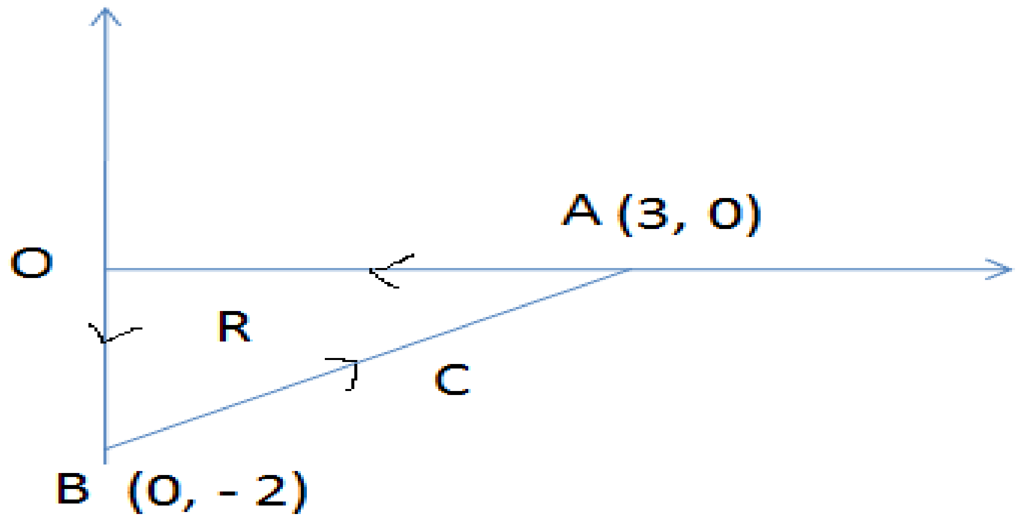
Example 2

State and verify Green's Theorem in the plane for $\oint (3x^2 - 8y^2)\, dx + (4y - 6xy)\; dy$ *where C is the boundary of the region bounded by* $x \geq 0,\ y \leq 0$ and $2x - 3y = 6$.

Solution

Here the closed curve C consists of straight lines OB, BA and AO, where coordinates of A and B are (3, 0) and (0, – 2) respectively. Let R be the region bounded by C.

Then by Green's Theorem in plane, we have



$$\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2) \right] dx\, dy \qquad \ldots(1)$$

$$= \iint_R (-6y + 16y)\, dx\, dy = \iint_R 10y\, dx\, dy$$

$$= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^{0} y\, dy = 10 \int_0^3 dx \left[\frac{y^2}{2} \right]_{\frac{1}{3}(2x-6)}^{0} = -\frac{5}{9} \int_0^3 dx\, (2x-6)^2$$

$$= -\frac{5}{9} \left[\frac{(2x-6)^3}{3 \times 2} \right]_0^3 = -\frac{5}{54}(0+6)^3 = -\frac{5}{54}(216) = -20 \quad ...(2)$$

Now we evaluate L.H.S. of (1) along OB, BA and AO.

Along OB, $x = 0$, $dx = 0$ and y varies form 0 to –2.

Along BA, $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2}\, dy$ and y varies from –2 to 0.

and along AO, $y = 0$, $dy = 0$ and x varies from 3 to 0.

L.H.S. of (1) $= \oint [\, (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$

$$= \int_{OB}[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy] + \int_{BA}[(3x^2 - 8y^2)\,dx + (4x - 6xy)\,dy]$$

$$+ \int_{AO}[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$$

$$= \int_0^{-2} 4y\,dy + \int_{-2}^{0}\left[\frac{3}{4}(6+3y)^2 - 8y^2\right]\left(\frac{3}{2}\,dy\right) + [4y - 3\,(6+3y)\,y]\,dy + \int_3^0 3x^2\,dx$$

$$= [2y^2]_0^{-2} + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right]dy + (x^3)_3^0$$

$$= 2\,[4] + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 21y^2 - 14y\right]dy + (0 - 27)$$

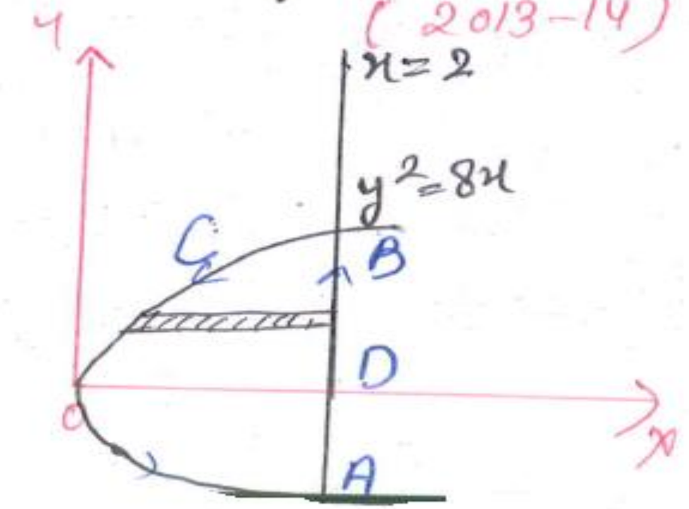
$$= 8 + \left[\frac{9}{8}\,\frac{(6+3y)^3}{3\times 3} - 7y^3 - 7y^2\right]_{-2}^{0} - 27 = -19 + \left[\frac{216}{8} + 7\,(-2)^3 + 7\,(-2)^2\right]$$

$$= -19 + 27 - 56 + 28 = -20 \qquad \ldots(3)$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

Q.2.1 Verify Green's theorem in plane for; $\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$, where C is the boundary of the region defined by $y^2=8x$ and $x=2$.

(2013-14)



Solⁿ:- Using Green's theorem

$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2y+3) - \frac{\partial}{\partial y}(x^2-2xy)\right] dx\,dy$$

$$= \iint_R (2xy+2x)\,dx\,dy = \int_{y=-4}^{4}\int_{x=y^2/8}^{2} (2xy+2x)\,dx\,dy$$

$$= \int_{y=-4}^{4} \left[x^2y + x^2\right]_{x=y^2/8}^{2} dy = \int_{y=-4}^{4} \left[4y+4-\frac{y^5}{64}-\frac{y^4}{64}\right] dy$$

$$= \left[2y^2+4y-\frac{y^6}{64\times 6}-\frac{y^5}{64\times 5}\right]_{-4}^{4} = \frac{128}{5}$$

Verification of the Green's theorem:

$$\oint_c (x^2-2xy)\,dx + (x^2y+3)\,dy = \left[\int_{BOA}[(x^2-2xy)\,dx + (x^2y+3)\,dy] + \int_{ADB}[(x^2-2xy)\,dx + (x^2y+3)\,dy]\right]$$

Along BOA, $y^2=8x$, $dx = \frac{y}{4}dy$

Along ADB, $x=2$, $dx=0$

$$= \int_{y=4}^{-4}\left[\left(\frac{y^4}{64}-\frac{y^3}{4}\right)\frac{y}{4}dy + \left(\frac{y^5}{64}+3\right)dy\right]$$

$$+ \int_{y=-4}^{4}[(4-4y)\cdot 0 + (4y+3)\,dy]$$

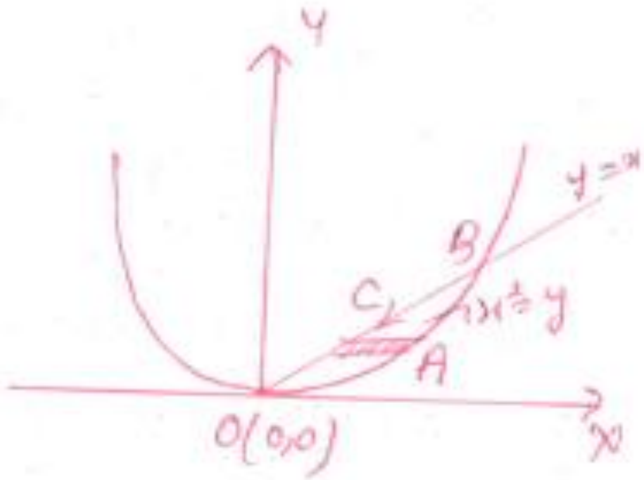
$$= \left[\frac{y^6}{64\times 24} - \frac{y^5}{16\times 5} + \frac{y^6}{64\times 6} + 3y\right]_4^{-4} + \left[2y^2+3y\right]_{-4}^{4}$$

$$= \frac{128}{5} - 24 + 24 = \frac{128}{5}$$

Verified

Q.3:1 Verify the Green's theorem to evaluate the line integral $\int_C (2y^2 dx + 3x dy)$, where C is the boundary of the closed region bounded by $y=x$ and $y=x^2$. (2015-16)

Sol^n:1 Using Green's theorem



$$\int_C 2y^2 dx + 3x dy = \iint \left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y^2)\right] dx\, dy$$

$$= \iint (3-4y)\, dx\, dy = \int_0^1 \int_{x=y}^{\sqrt{y}} (3-4y)\, dx\, dy$$

$$= \int_0^1 [3x - 4xy]_y^{\sqrt{y}}\, dy = \int_0^1 (3\sqrt{y} - 4y^{3/2} - 3y + 4y^2)\, dy$$

$$= \left[3 \times \frac{2}{3} y^{3/2} - 4 \times \frac{2}{5} y^{5/2} - \frac{3y^2}{2} + \frac{4y^3}{3}\right]_0^1 = \frac{6}{3} - \frac{8}{5} - \frac{3}{2} + \frac{4}{3} = \frac{7}{30}$$

Verification of the Green's theorem:

$$\int_C 2y^2dx + 3x\,dy = \int_{OAB} (2y^2\,dx + 3x\,dy) + \int_{BCO} (2y^2dx + 3x\,dy)$$

Along OAB, $y = x^2$, $dy = 2x\,dx$

Along BCO, $y = x$, $dy = dx$

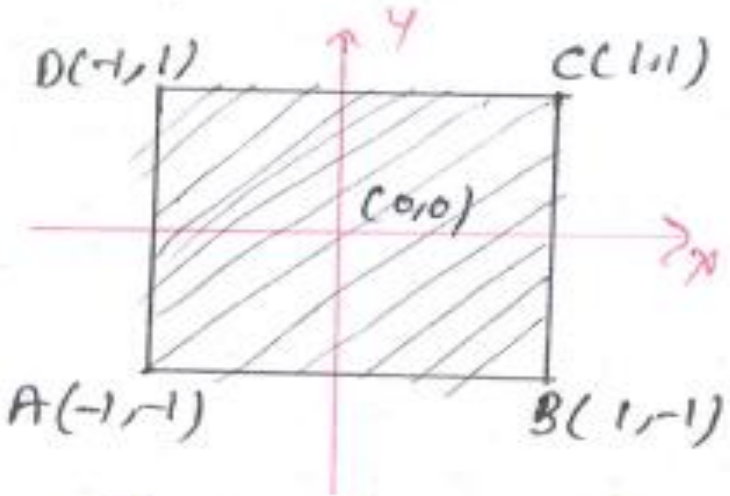
$$= \int_0^1 [2x^4 + 3x(2x)]\,dx + \int_1^0 (2x^2 + 3x)\,dx$$

$$= \left[\frac{2x^5}{5} + 2x^3\right]_0^1 + \left[\frac{2x^3}{3} + \frac{3x^2}{2}\right]_1^0$$

$$= \left(\frac{2}{5} + 2\right) - \left(\frac{2}{3} + \frac{3}{2}\right) = \frac{12}{5} - \frac{13}{6} = \frac{72-65}{30} = \frac{7}{30}$$ Verified

Q.4: Verify Green's theorem, evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$, where C square formed by lines $x=\pm 1, y=\pm 1$. (2017-18)

Sol^n: By Green's theorem, we have



$$\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2+xy)\right] dx\,dy$$

$$= \iint_R (2x-x)\,dx\,dy = \int_{x=-1}^{1}\int_{y=-1}^{1} x\,dx\,dy = \int_{-1}^{1} x\,(y)_{-1}^{1}\,dx = \int_{-1}^{1} 2x\,dx$$
$$= 0$$

Verification of the Green's theorem:

$$\int_C [(x^2+xy)\,dx + (x^2+y^2)\,dy] = \int_{AB} [(x^2+xy)\,dx + (x^2+y^2)\,dy] +$$

$$\int_{BC}[(x^2+xy)dx+(x^2+y^2)dy] + \int_{CD}[(x^2+xy)dx+(x^2+y^2)dy]$$

$$+\int_{DA}[(x^2+xy)dx+(x^2+y^2)dy]$$

Now along AB, $y=-1$ and $dy=0$

$$\therefore \int_{AB}[(x^2+xy)dx+(x^2+y^2)dy] = \int_{-1}^{1}(x^2-x)dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left[\frac{1}{3}-\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\right] = \frac{2}{3}$$

Along BC, $x=1$ and $dx=0$

$$\therefore \int_{BC}[(x^2+xy)dx+(x^2+y^2)dy] = \int_{-1}^{1}(1+y^2)dy = \left[y+\frac{y^3}{3}\right]_{-1}^{1} = \frac{8}{3}$$

Along CD, $y=1$ and $dy=0$

$$\therefore \int_{CD}\left[(x^2+xy)\,dx+(x^2+y^2)\,dy\right] = \int_{-1}^{1}(x^2+x)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left[\frac{-1}{3}+\frac{1}{2}-\frac{1}{3}-\frac{1}{2}\right] = -\frac{2}{3}$$

Along DA, $x=-1$ and $dx=0$

$$\therefore \int_{DA}\left[(x^2+xy)\,dx+(x^2+y^2)\,dy\right] = \int_{+1}^{-1}(1+y^2)\,dy = \left[y+\frac{y^3}{3}\right]_{1}^{-1}$$

$$= \left[-1-\frac{1}{3}-1-\frac{1}{3}\right] = -\frac{8}{3}$$

$$\Rightarrow \int_{C}\left[(x^2+xy)\,dx+(x^2+y^2)\,dy\right] = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0 \quad \text{Verified}$$

Practice Questions

1 Verify Green's Theorem in plane for $\int_C (x^2 + 2xy)\,dx + (y^2 + x^3 y)\,dy$, where c is a square with the vertices P (0, 0), Q (1, 0), R (1, 1) and S (0, 1). **Ans.** $-\frac{1}{2}$

2 Verify Green's Theorem for $\int_C \left[(xy + y^2)\,dx + x^2 dy\right]$ where C is the boundary by $y = x$ and $y = x^2$.

3 Verify Green's Theorem for $\int_c (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy$ around the boundary c of the region $y^2 = 8x$ and $x = 2$.

4 Verify the Green's Theorem to evaluate the line integral $\int_c (2y^2\,dx + 3x\,dy)$, where c is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

Ans. $\frac{27}{4}$

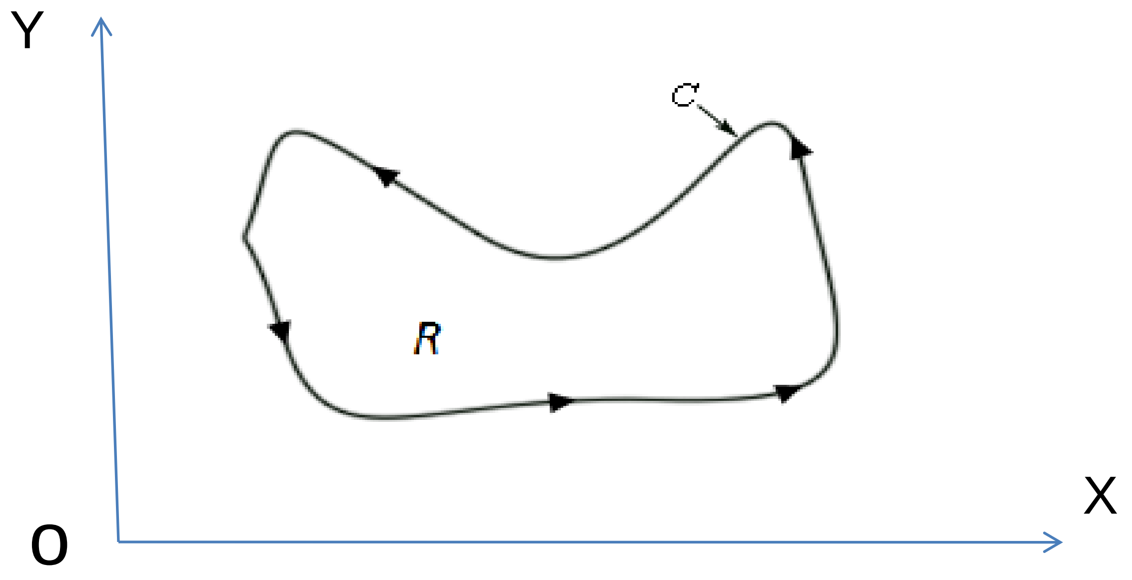
Lecture 41(II)

Green's Theorem and its Applications - II

Green Theorem

If $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial\phi}{\partial v}$ and $\frac{\partial\psi}{\partial x}$ *be continuous functions over a region R bounded by simple closed curve C in* $x - y$ *plane, then*

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$



Area of Plane Region by Green Theorem

We know that

$$\int_C M dx + N dy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad \ldots(1)$$

On putting $N = x \left(\frac{\partial N}{\partial x} = 1 \right)$ and $M = -y \left(\frac{\partial M}{\partial y} = 1 \right)$ in (1), we get

$$\int_C -y\, dx + x\, dy = \iint_A [1 - (-1)]\, dx\, dy = 2 \iint dx\, dy = 2A$$

$$\text{Area} = \frac{1}{2} \int_C (x\, dy - y\, dx)$$

Example 3

Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}$$

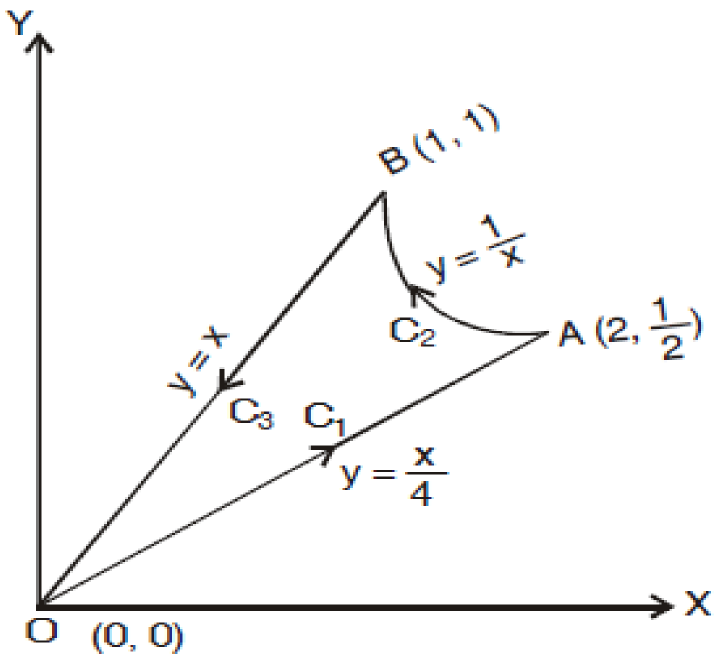
Solution

By Green's Theorem Area A of the region bounded by a closed curve C is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

Here, C consists of the curves $C_1 : y = \frac{x}{4}$, $C_2 : y = \frac{1}{x}$ and $C_3 : y = x$ So

$$\left[A = \frac{1}{2}\oint_C = \frac{1}{2}\left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] = \frac{1}{2}(I_1 + I_2 + I_3)\right]$$



Along $\quad C_1 : y = \frac{x}{4},\ dy = \frac{1}{4}dx,\ x : 0 \text{ to } 2$

$$I_1 = \int_{C_1}(xdy - ydx) = \int_{C_1}\left(x\frac{1}{4}dx - \frac{x}{4}dx\right) = 0$$

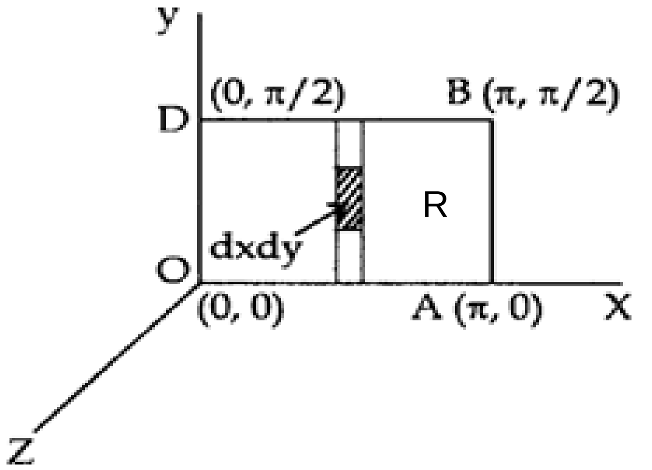
Along $\quad C_2 : y = \frac{1}{x},\ dy = -\frac{1}{x^2}dx,\ x : 2 \text{ to } 1$

$$I_2 = \int_{C_2}(xdy - ydx) = \int_2^1\left[x\left(-\frac{1}{x^2}\right)dx - \frac{1}{2}dx\right] = [-2\log x]_2^1 = 2\log 2$$

Example 1

Evaluate by Green's theorem $\int_C [e^{-x} \sin y dx + e^{-x} \cos y dy]$ where C is the rectangle with vertices (0,0), (π,0), (π, π/2), (0, π/2) and hence verify Green's theorem.

Solution



By Green's theorem we have

$$\int_C (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Comparing the given integral

$M = e^{-x} \sin y$ and $N = e^{-x} \cos y$

Then $\frac{\partial M}{\partial y} = e^{-x} \cos y$ and $\frac{\partial N}{\partial x} = -e^{-x} \cos y$

Hence by Green's theorem

$$\int_C [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] = \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_{x=0}^{\pi}\int_{y=0}^{\pi/2} e^{-x}\cos y\,dx\,dy$$

$$= -2\left[-e^{-x}\right]_0^{\pi}\left[\sin y\right]_0^{\pi/2}$$

$$= 2(e^{-\pi}-1)(1)$$

$$= 2(e^{-\pi}-1)$$

Evaluation of line integral:

For this purpose, let us evaluate the given line integral directly.

$$\int_C [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy]$$

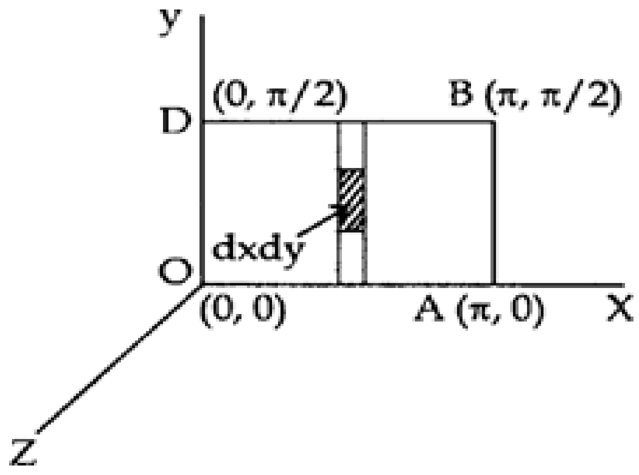
$$= \int_{OA} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] + \int_{AB} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] + \int_{BD} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy] + \int_{DO} [e^{-x}\sin y\,dx + e^{-x}\cos y\,dy]$$

Now along OA, $y = 0 \Rightarrow dy = 0$

along AB, $x = \pi \Rightarrow dx = 0$

along BD, $y = \pi/2 \Rightarrow dy = 0$

along DO, $x = 0 \Rightarrow dx = 0$



Hence the given line integral

$$= 0 + \int_0^{\pi/2} e^{-\pi} \cos y \, dy + \int_{\pi}^{0} e^{-x} dx + \int_{\pi/2}^{0} \cos y \, dy$$

$$= e^{-\pi} [\sin y]_0^{\pi/2} + \left[-e^{-x}\right]_{\pi}^{0} + [\sin y]_{\pi/2}^{0}$$

$$= e^{-\pi} - (1 - e^{-\pi}) + (-1) = 2(e^{-\pi} - 1)$$

Hence Green's theorem is verified.

Example 2

State and verify Green's Theorem in the plane for $\oint (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ *where C is the boundary of the region bounded by* $x \geq 0,\ y \leq 0$ and $2x - 3y = 6$.

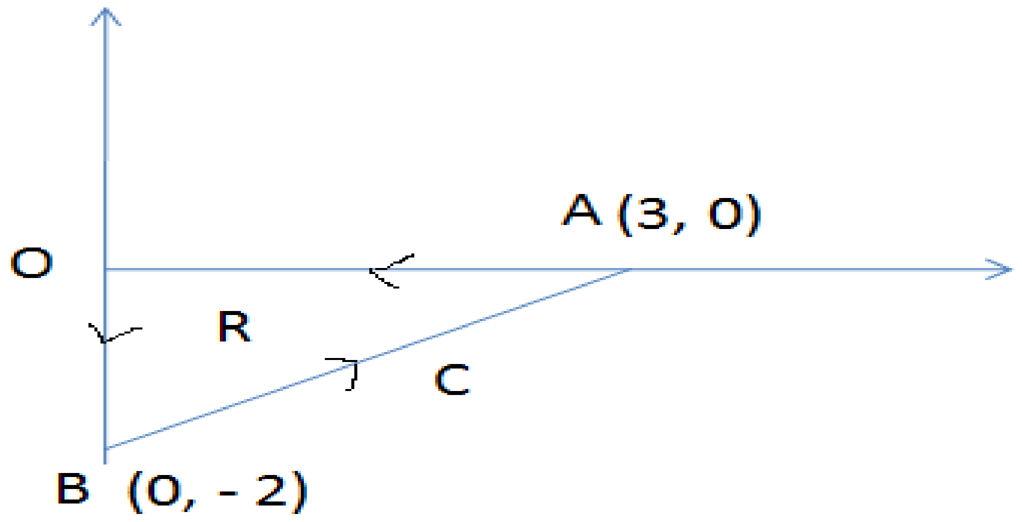
Solution

Here the closed curve C consists of straight lines OB, BA and AO, where coordinates of A and B are (3, 0) and (0, – 2) respectively. Let R be the region bounded by C.

Then by Green's Theorem in plane, we have

$$\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2) \right] dx\, dy \qquad \ldots(1)$$



$$= \iint_R (-6y + 16y)\, dx\, dy = \iint_R 10y\, dx\, dy$$

$$= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^{0} y\, dy = 10 \int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{1}{3}(2x-6)}^{0} = -\frac{5}{9}\int_0^3 dx\, (2x-6)^2$$

$$= -\frac{5}{9}\left[\frac{(2x-6)^3}{3\times 2}\right]_0^3 = -\frac{5}{54}(0+6)^3 = -\frac{5}{54}(216) = -20 \quad ...(2)$$

Now we evaluate L.H.S. of (1) along OB, BA and AO.

Along OB, $x = 0$, $dx = 0$ and y varies form 0 to –2.

Along BA, $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2}\, dy$ and y varies from –2 to 0.

and along AO, $y = 0$, $dy = 0$ and x varies from 3 to 0.

L.H.S. of (1) = $\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$

$$= \int_{OB}[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy] + \int_{BA}[(3x^2 - 8y^2)\,dx + (4x - 6xy)\,dy]$$

$$+ \int_{AO}[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$$

$$= \int_0^{-2} 4y\,dy + \int_{-2}^{0}\left[\frac{3}{4}(6+3y)^2 - 8y^2\right]\left(\frac{3}{2}\,dy\right) + [4y - 3\,(6+3y)\,y]\,dy + \int_3^0 3x^2\,dx$$

$$= [2y^2]_0^{-2} + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right]dy + (x^3)_3^0$$

$$= 2\,[4] + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 21y^2 - 14y\right]dy + (0 - 27)$$

$$= 8 + \left[\frac{9}{8}\,\frac{(6+3y)^3}{3\times 3} - 7y^3 - 7y^2\right]_{-2}^{0} - 27 = -19 + \left[\frac{216}{8} + 7\,(-2)^3 + 7\,(-2)^2\right]$$

$$= -19 + 27 - 56 + 28 = -20 \qquad \ldots(3)$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

Practice Questions

1 Apply Green's Theoem to evaluate $\int_c [(y - \sin x)\, dy + \cos x\, dy]$, where c is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$. **Ans.** $-\frac{\pi^2 + 8}{4\pi}$

2 Use Green's Theorem in a plane to evaluate the integral $\int_c [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, where c is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle $x^2 + y^2 = 1$ in the upper half xy-plane. **Ans.** $\frac{4}{3}$

3 Green's Therorem, evaluate the line integral $\int_c e^{-x} (\cos y\, dx - \sin y\, dy)$, where c is the rectangle with vertices $(0, 0)$, $(\pi, 0,)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$. **Ans.** $2\,(1 - e^{-\pi})$

4 *Use Green's theorem to evaluate* $\int_C (x^2 + xy)\, dx + (x^2 + y^2)\, dy$ *where C is the square formed by the lines* $y = \pm 1$, $x = \pm 1$.

Ans: 0

Practice Questions

1 Verify Green's Theorem in plane for $\int_C (x^2 + 2xy)\,dx + (y^2 + x^3 y)\,dy$, where c is a square with the vertices P (0, 0), Q (1, 0), R (1, 1) and S (0, 1). **Ans.** $-\frac{1}{2}$

2 Verify Green's Theorem for $\int_C \left[(xy + y^2)\,dx + x^2 dy\right]$ where C is the boundary by $y = x$ and $y = x^2$.

3 Verify Green's Theorem for $\int_c (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy$ around the boundary c of the region $y^2 = 8x$ and $x = 2$.

4 Verify the Green's Theorem to evaluate the line integral $\int_c (2y^2\,dx + 3x\,dy)$, where c is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

Ans. $\frac{27}{4}$

Lecture 42(I)

Stoke's Theorem and It's Applications - I

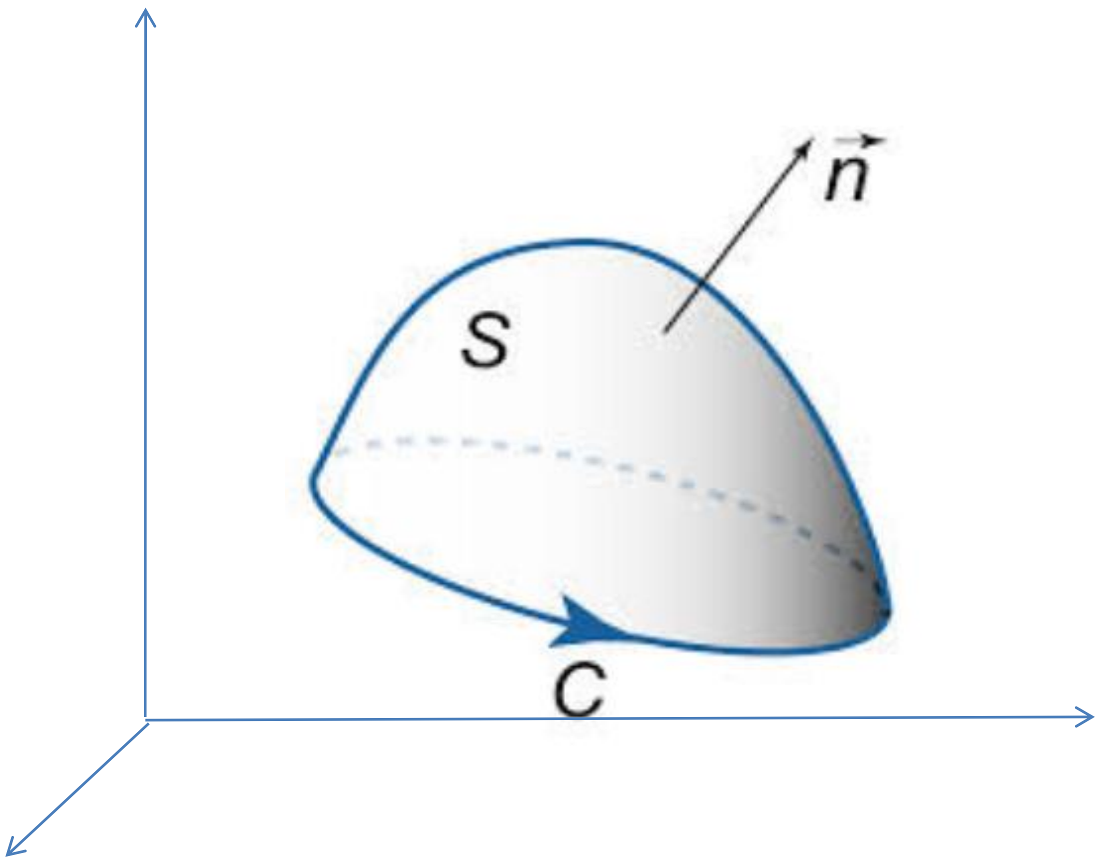
Stoke's Theorem

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically

$$\oint \vec{F} \cdot d\,\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ is a unit external normal to the surface.



Stokes' Theorem relates a surface integral over an open surface S to a line integral around the boundary curve of S (a space curve).

Example 1

Using Stoke's theorem or otherwise, evaluate

$$\int_C [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

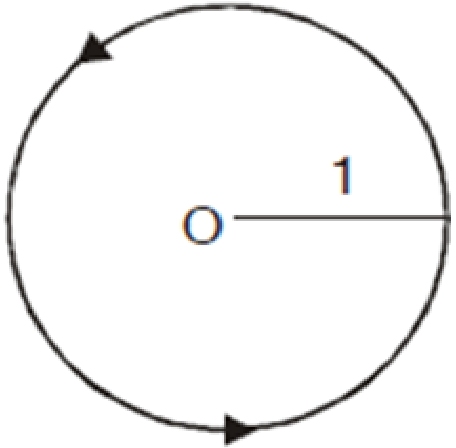
where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

Solution

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

$$= \int_c [(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}] \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

By Stoke's theorem $\oint \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n}\,ds$



$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2\,yz + 2\,yz)\,\hat{i} - (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}$$

Putting the value of curl $\vec{F}$ in (1), we get

$$= \iint \hat{k} \cdot \hat{n}\, ds = \iint \hat{k} \cdot \hat{n} \frac{dx\,dy}{\hat{n} \cdot \hat{k}} = \iint dx\,dy = \text{Area of the circle} = \pi \qquad \left[\because ds = \frac{dx\,dy}{(\hat{n} \cdot \hat{k})}\right]$$

Example 2

Verify Stoke's Theorem for the function

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane z = 0 and bounded by the lines

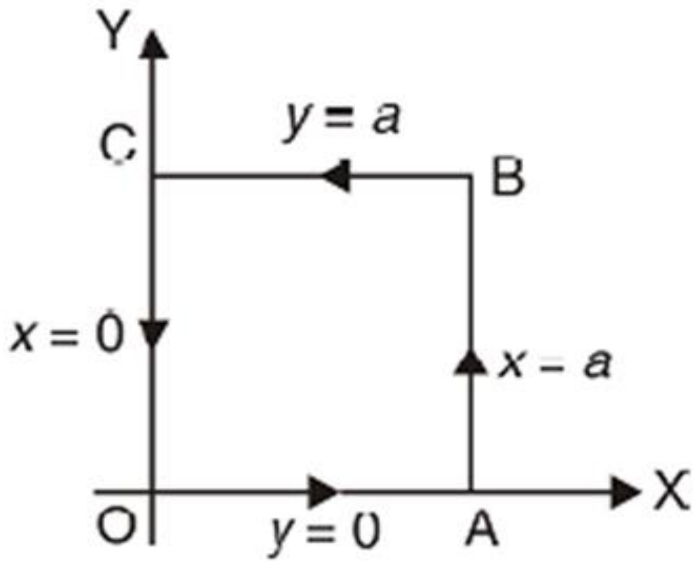
$$x = 0,\ y = 0,\ x = a,\ y = a.$$

Solution

We have, $\vec{F} = x^2\hat{i} - xy\hat{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= (0-0)\,\hat{i} - (0-0)\,\hat{j} + (-y-0)\,\hat{k} = -y\hat{k}$$



($\hat{n} \perp$ to xy plane *i.e.* $\hat{k}$)

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S (-yk) \cdot k\, dx\, dy$$

$$= \int_0^a dx \int_0^a -y dy = \int_0^a dx \left[-\frac{y^2}{2} \right]_0^a = -\frac{a^2}{2} (x)_0^a = -\frac{a^3}{2} \quad \text{...(1)}$$

To obtain line integral

$$\int_C \vec{F} \cdot \vec{d}r = \int_C (x^2 \hat{i} - xy\hat{j}) \cdot (\hat{i}\, dx + \hat{j}\, dy) = \int_C (x^2\, dx - xy\, dy)$$

where c is the path $OABCO$ as shown in the figure.

Also, $$\int_C \vec{F} \cdot \vec{d}r = \int_{OABCO} \vec{F} \cdot dr = \int_{OA} \vec{F} \cdot dr + \int_{AB} \vec{F} \cdot dr + \int_{BC} \vec{F} \cdot dr + \int_{CO} \vec{F} \cdot dr \quad \text{...(2)}$$

Along OA, $y = 0$, $dy = 0$

$$\int_{OA} \vec{F} \cdot \vec{dr} = \int_{OA} (x^2 dx - xy dy)$$

$$= \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along AB, $x = a$, $dx = 0$

$$\int_{AB} \vec{F} \cdot \vec{dr} = \int_{AB} (x^2\, dx - x\, y\, d\, y)$$

$$= \int_0^a - a\, y\, d\, y = -a\left[\frac{y^2}{2}\right]_0^a = -\frac{a^3}{2}$$

line	Eq. of line		Lower limit	Upper limit
OA	$y = 0$	$dy = 0$	$x = 0$	$x = a$
AB	$x = a$	$dx = 0$	$y = 0$	$y = a$
BC	$y = a$	$dy = 0$	$x = a$	$x = 0$
CO	$x = 0$	$dx = 0$	$y = a$	$y = 0$

Along BC, $y = a,\ dy = 0$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_{BC} (x^2 dx - xy\, dy) = \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

Along CO, $x = 0,\ dx = 0$

$$\int_{CO} \vec{F} \cdot \vec{dr} = \int_{CO} (x^2 dx - xy\, dy) = 0$$

Putting the values of these integrals in (2), we have

$$\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} - \frac{a^3}{2} - \frac{a^3}{3} + 0 = -\frac{a^3}{2} \qquad \ldots(3)$$

From (1) and (3), $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \int_C \vec{F} \cdot \vec{dr}$

Hence, Stoke's Theorem is verified. **Ans.**

Example 3

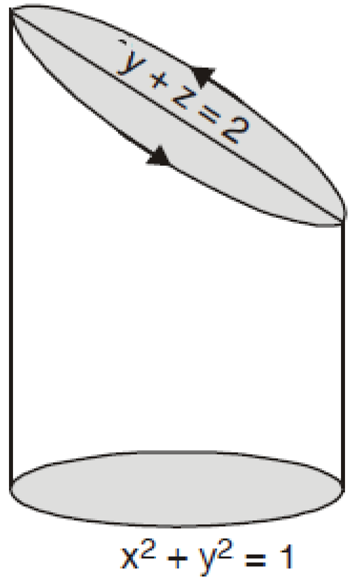
Evaluate $\int_C \vec{F} \cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$.

Solution

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds = \iint_S \text{curl}\,(-y^2\hat{i} + x\hat{j} + z^2\hat{k}) \cdot \hat{n}\, ds \qquad \ldots(1)$$

$$F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

$$\text{Curl}\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+2y) = (1+2y)\hat{k}$$



Normal vector $= \nabla \vec{F}$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (y + z - 2) = \hat{j} + \hat{k}$$

Unit normal vector $\hat{n} \quad = \dfrac{\hat{j} + \hat{k}}{\sqrt{2}}$

$$ds = \frac{dx\, dy}{\hat{n} \,.\, \hat{k}}$$

On putting the values of curl $\vec{F}, \hat{n}$ and ds in (1), we get

$$\int_C \vec{F} \,.\, \vec{dr} = \iint_S (1 + 2y)\, \hat{k} \,.\, \frac{\hat{j} + \hat{k}}{\sqrt{2}} \frac{dx\, dy}{\left(\frac{\hat{j} + \hat{k}}{\sqrt{2}} \right) . \hat{k}}$$

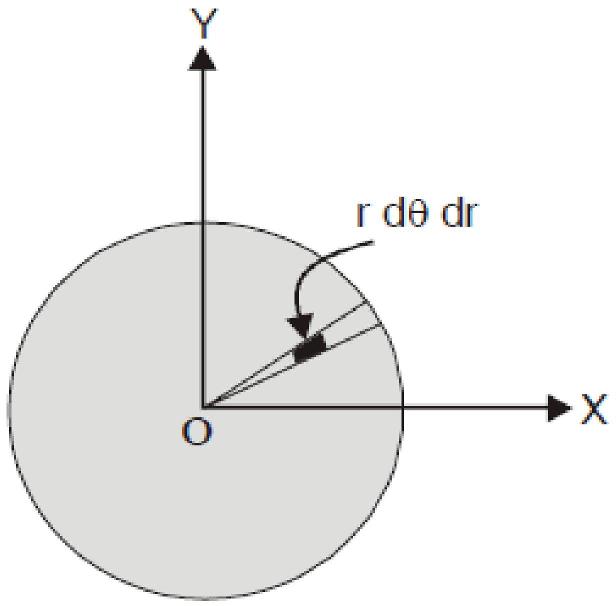
$$= \iint \frac{1 + 2y}{\sqrt{2}} \frac{dx\, dy}{\frac{1}{\sqrt{2}}} = \iint (1 + 2y)\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta)\, r\, d\theta\, dr$$

$$= \int_0^{2\pi} \int_0^1 (r + 2r^2 \sin\theta)\, d\theta\, dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\sin\theta\right] d\theta$$

$$= \left[\frac{\theta}{2} - \frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi - \frac{2}{3} - 0 + \frac{2}{3}\right) = \pi$$



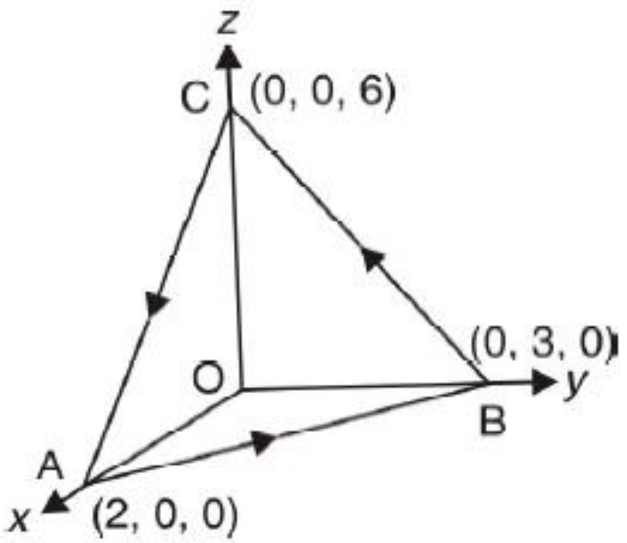
Example 3

Verify Stoke's Theorem for $\vec{F} = (x + y)\,\hat{i} + (2x - z)\,\hat{j} + (y + z)\,\hat{k}$ *for the surface of a triangular lamina with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

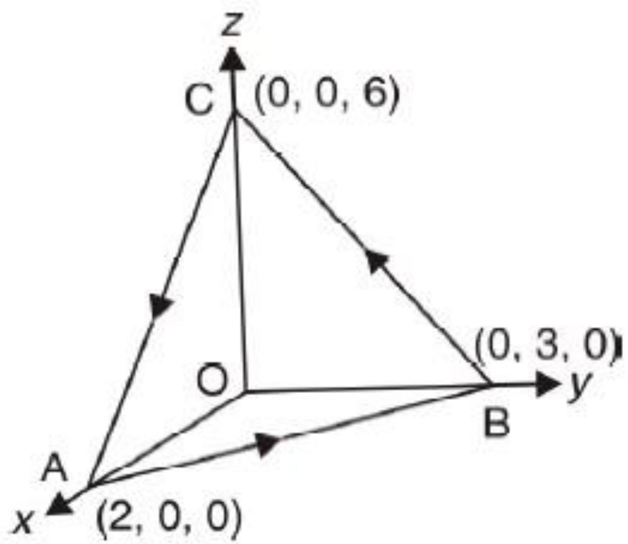
Solution

Here the path of integration c consists of the straight lines AB, BC, CA where the co-ordinates of A, B, C and (2, 0, 0), (0, 3, 0) and (0, 0, 6) respectively. Let S be the plane surface of triangle ABC bounded by C. Let $\hat{n}$ be unit normal vector to surface S. Then by Stoke's Theorem, we must have

$$\oint_c \vec{F} \cdot \vec{dr} = \iint_s \text{curl } \vec{F} \cdot \hat{n}\, ds \qquad \ldots(1)$$



L.H.S. of (1)= $\int_{ABC} \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CA} \vec{F}\cdot\vec{dr}$



Along line AB, $z = 0$, equation of AB is $\frac{x}{2} + \frac{y}{3} = 1$

$$\Rightarrow \quad y = \frac{3}{2}(2 - x),\ dy = -\frac{3}{2}dx$$

At A, $x = 2$, At B, $x = 0$, $\bar{r} = x\hat{i} + y\hat{j}$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{AB}[(x+y)\hat{i} + 2x\hat{j} + y\hat{k}]\cdot(\hat{i}dx + \hat{j}dy)$$

$$= \int_{AB}(x+y)\,dx + 2xdy$$

$$= \int_{AB}\left(x + 3 - \frac{3x}{2}\right)dx + 2x\left(-\frac{3}{2}dx\right)$$

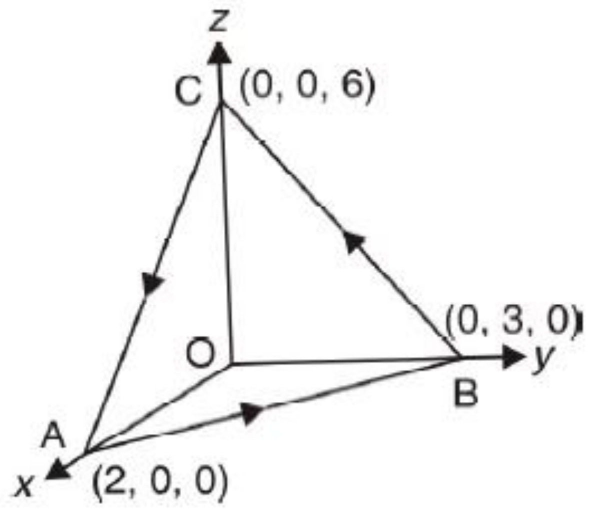
$$= \int_2^0\left(-\frac{7x}{2} + 3\right)dx = \left(-\frac{7x^2}{4} + 3x\right)_2^0$$

$$= (7 - 6) = +1$$

line	Eq. of line		Lower limit	Upper limit
AB	$\frac{x}{2} + \frac{y}{3} = 1$, $z = 0$	$dy = -\frac{3}{2}dx$	At A $x = 2$	At B $x = 0$

Along line BC, $x = 0$, Equation of BC is $\frac{y}{3} + \frac{z}{6} = 1$ or $z = 6 - 2y$, $dz = -2dy$

At B, $y = 3$, At C, $y = 0$, $\vec{r} = y\hat{j} + z\hat{k}$



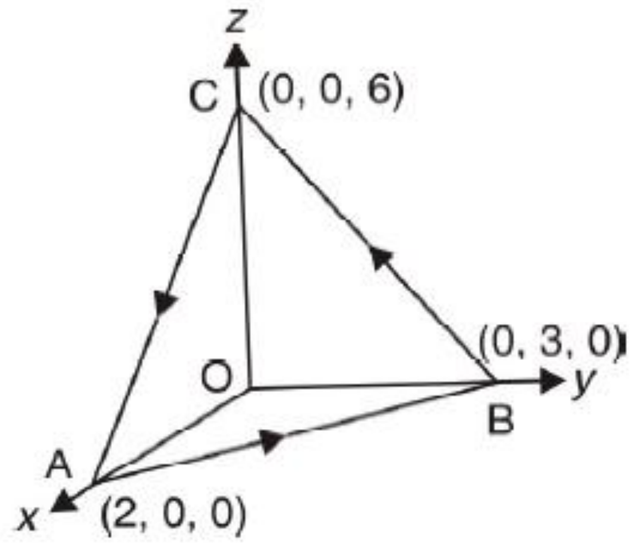
$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_{BC} [yi + zj + (y + z)k] \cdot (jdy + kdz) = \int_{BC} -zdy + (y + z)\, dz$$

$$= \int_3^0 (-6 + 2y)\, dy + (y + 6 - 2y)(-2dy)$$

$$= \int_3^0 (4y - 18)\, dy = (2y^2 - 18y)_3^0 = 36$$

line	Eq. of line		Lower limit	Upper limit
BC	$\frac{y}{3} + \frac{z}{6} = 1$ $x = 0$	$dz = -2dy$	At B $y = 3$	At C $y = 0$

line	Eq. of line		Lower limit	Upper limit
CA	$\frac{x}{2}+\frac{z}{6}=1$, $y=0$	$dz=-3dx$	At C $x=0$	At A $x=2$



Along line CA, $y = 0$, Eq. of CA, $\frac{x}{2}+\frac{z}{6}=1$ or $z = 6 - 3x$, $dz = -3dx$

At C, $x = 0$, at A, $x = 2$, $\vec{r} = x\hat{i} + z\hat{k}$

$$\int_{CA} \vec{F}\cdot\vec{dr} = \int_{CA}[x\hat{i} + (2x - z)\,\hat{j} + z\hat{k}]\cdot[dx\hat{i} + dz\hat{k}] = \int_{CA}(xdx + zdz)$$

$$= \int_0^2 xdx + (6 - 3x)\,(-3dx) = \int_0^2 (10x - 18)\,dx = [5x^2 - 18x]_0^2 = -16$$

Lecture 42(II)

Stoke's Theorem and It's Applications - II

Q.1: Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where:
$\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$.

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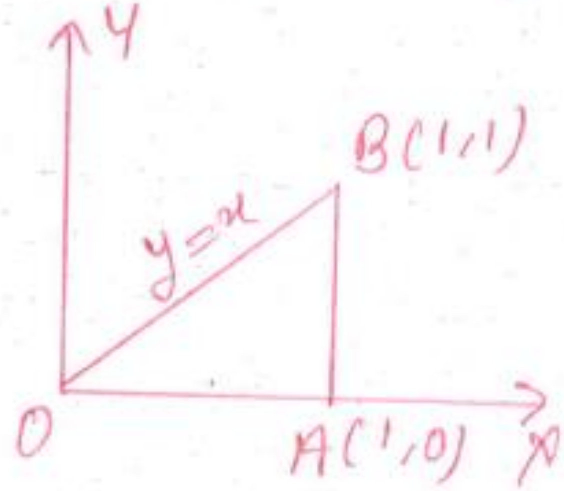

Sol^n: Since z-coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy-plane and $\hat{n} = \hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$

$$\therefore \text{curl}\,\vec{F}\cdot\hat{n} = [\hat{j} + 2(x-y)\hat{k}]\cdot\hat{k} = 2(x-y)$$

The equation of line OB is $y = x$.

By Stoke's theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^x 2(x-y)\,dy\,dx = \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_0^1\left(x^2 - \frac{x^2}{2}\right)dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

Q.2:→ Verify Stokes theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$.

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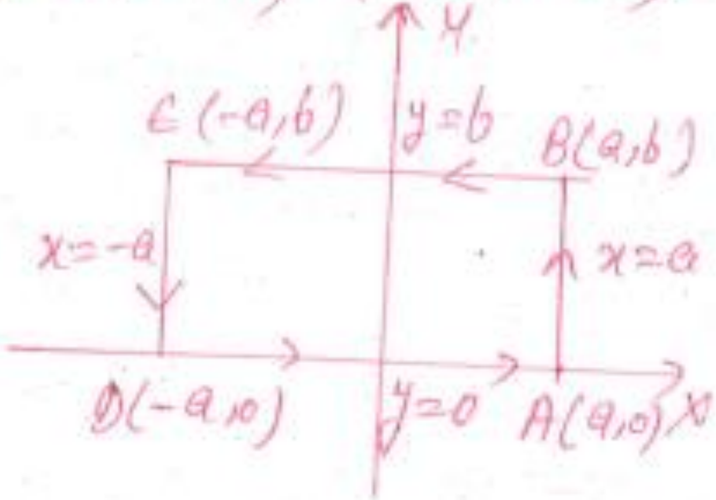
Sol^n:→ Let C denote the boundary of the rectangle ABED, then



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(x^2+y^2)\hat{i} - 2xy\hat{j}](\hat{i}dx + \hat{j}dy)$$

$$= \oint_C [(x^2+y^2)dx - 2xy\,dy]$$

The curve C consists of four lines AB, BE, ED and DA.

Along AB, $x = a$, $dx = 0$ and y varies from 0 to b

$$\therefore \int_{AB} [(x^2+y^2)dx - 2xy\,dy] = \int_0^b -2ay\,dy = -a[y^2]_0^b = -ab^2 \quad \text{--- (1)}$$

Along BE, $y=b$, $dy=0$ and x varies from a to $-a$.

$$\therefore \int_{BE}[(x^2+y^2)dx - 2xy\,dy] = \int_a^{-a}(x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_a^{-a}$$

$$= \frac{-2a^3}{3} - 2ab^2 \quad \text{——— (2)}$$

Along ED, $x=-a$, $dx=0$ and y varies from b to 0.

$$\therefore \int_{ED}[(x^2+y^2)dx - 2xy\,dy] = \int_b^0 2ay\,dy = a[y^2]_b^0 = -ab^2 \quad \text{——— (3)}$$

Along DA, $y=0$, $dy=0$ and x varies from $-a$ to a.

$$\therefore \int_{DA}[(x^2+y^2)dx - 2dx\,dy] = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3} \quad \text{——— (4)}$$

Adding (1), (2), (3) and (4), we get

$$\oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \quad \text{——— (5)}$$

Now $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\hat{k} = -4y\hat{k}$

For the surface S, $\hat{n} = \hat{k}$

$$\text{curl}\,\vec{F}\cdot\hat{n} = -4y\hat{k}\cdot\hat{k} = -4y$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS = \int_0^b\int_{-a}^{a} -4y\, dx\, dy = \int_0^b -4y\,[x]_{-a}^{a}\, dy$$

$$= -8a\int_0^b y\, dy = -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad \text{— (6)}$$

The equality of (5) and (6) verifies Stoke's theorem.

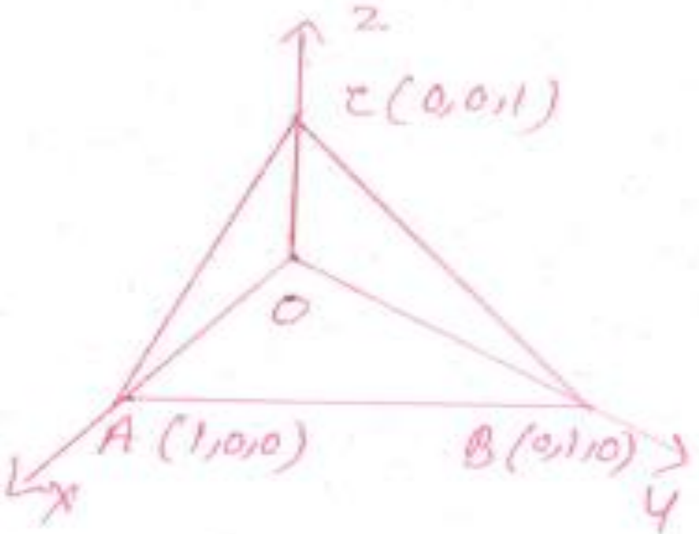
Q.3: Verify Stokes theorem $\vec{F} = (2y+z, x-z, y-x)$ taken over the triangle ABC cut from the plane $x+y+z=1$ by the coordinate planes. (2016-17)

Sol^n: By Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \qquad \text{—— (1)}$$

Taking LHS,

$$\oint_{ABC} \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r}$$



Along AB, $z=0$, $x+y=1$, $y=1-x$, $dy=-dx$

and $\vec{r} = x\hat{i} + y\hat{j}$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} [2y\hat{i} + x\hat{j} + (y-x)\hat{k}](\hat{i}\,dx + \hat{j}\,dy)$$

$$= \int_{AB} 2y\,dx + x\,dy = \int_{AB} 2(1-x)\,dx + x(-dx)$$

$[\because y = 1-x \text{ and } dy = -dx]$

$$= \int_{AB} 2\,dx - 2x\,dx - x\,dx = \int_1^0 (2-3x)\,dx = \left[2x - \frac{3x^2}{2}\right]_1^0 = -\frac{1}{2}$$

Along BC, $x = 0$, $y + z = 1$, $z = 1 - y$, $dz = -dy$ and $\vec{r} = y\hat{j} + z\hat{k}$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{BC} [(2y+z)\hat{i} - z\hat{j} + y\hat{k}]\,(\hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_{BC} -z\,dy + y\,dz = \int_{BC} -(1-y)\,dy + y(-dy) = \int_{BC} -dy + y\,dy - y\,dy$$

$$= \int_1^0 -dy = [-y]_1^0 = 0 - (-1) = 1.$$

Along CA, $y = 0$, $x + z = 1 \Rightarrow x = 1 - z$, $dx = -dz$ and $\vec{r} = x\hat{i} + z\hat{k}$

$$\int_{CA} \vec{F}\cdot d\vec{r} = \int_{CA} [z\hat{i} + (x-z)\hat{j} - x\hat{k}]\,(\hat{i}\,dx + \hat{k}\,dz) = \int_{CA} z\,dx - x\,dz$$

$$= \int_{CA} z(-dz) - (1-z)\,dz = \int_{AC} -z\,dz - dz + z\,dz = \int_{AC} -dz$$

$$= \int_{1}^{0} -dz = -[z]_1^0 = -[0-1] = 1$$

Hence $\int_{AB} \vec{F}\cdot d\vec{r} = \frac{-1}{2} + 1 + 1 = \frac{3}{2}$

Now, curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x-z & y-x \end{vmatrix}$

$$= \hat{i}\,[1-(-1)] - \hat{j}\,[-1-1] + \hat{k}\,[1-2] = 2\hat{i} + 2\hat{j} - \hat{k}$$

Equation of the plane ABC is $x+y+z=1$

Normal to the plane ABC is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y+z-1) = \hat{i}+\hat{j}+\hat{k}$$

Normal unit vector, $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Now taking RHS of eq^n ①

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S (2\hat{i}+2\hat{j}-\hat{k})\left(\cdot\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}} \quad \frac{dx\,dy}{\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\hat{k}}\right)$$

$$= \iint_S \frac{2+2-1}{\sqrt{3}}\;\frac{dx\,dy}{1/\sqrt{3}} = 3\iint dx\,dy = 3\times \text{area of } \Delta AOB$$

$$= 3\times\frac{1}{2}\times 1\times 1 = \frac{3}{2}$$ Verified

Q.4:- Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ integrated round the rectangle in the plane $z = 0$. and bounded by the lines $x = 0, y = 0, x = a, y = b$. (2019-20)

Soln:- Proceed as Q.2.

Q.5:- Verify Stoke's theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x = 0, y = 0, x = a, y = a$ in the plane $z = 0$. (2020-21)

Soln:- Proceed as Q.2.

Practice Questions

Q1 *Verify Stoke's Theorem for* $\vec{F} = (x + y)\,\hat{i} + (2x - z)\,\hat{j} + (y + z)\,\hat{k}$ *for the surface of a triangular lamina with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

Ans: $\iint_S Curl\,\vec{F}.\hat{n}\,dS = \int_C \vec{F}.\overrightarrow{dr} = 21.$

Q2 *Verify Stoke's Theorem for*

$$\vec{F} = (y - z + 2)\,\hat{i} + (yz + 4)\,\hat{j} - (xz)\,\hat{k}$$

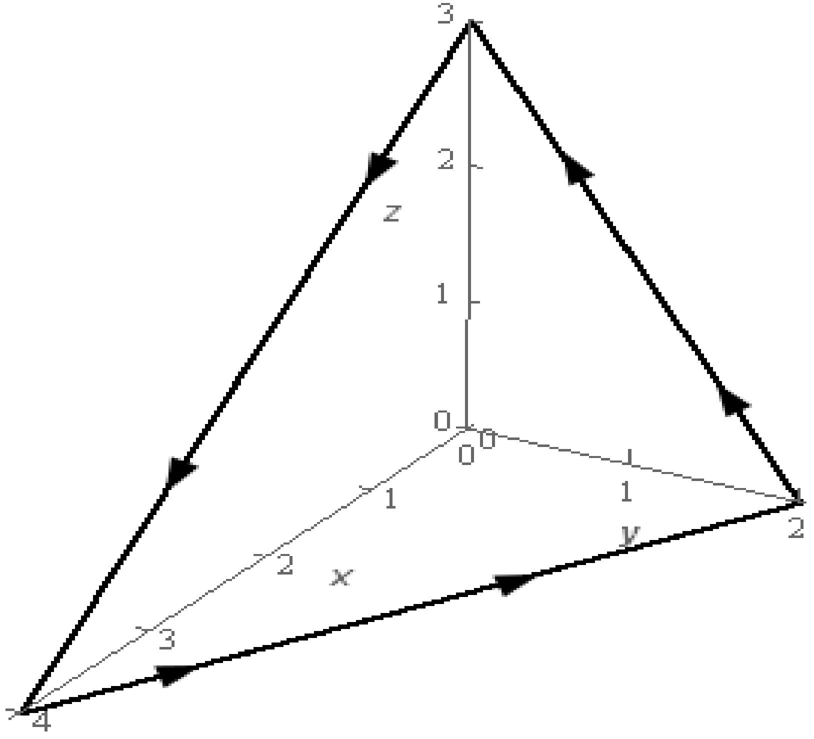
over the surface of a cube $x = 0,\ y = 0,\ z = 0,\ x = 2,\ y = 2,\ z = 2$ *above the XOY plane (open the bottom).*

Ans: - 4

Q3. *Use Stokes' Theorem to evaluate* $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (3yx^2 + z^3)\,\vec{i} + y^2\,\vec{j} + 4yx^2\,\vec{k}$ and C

is triangle with vertices (0,0,3), (0,2,0) and (4,0,0). C has a counter clockwise rotation if you are above the triangle and looking down towards the xy-plane.See the figure below for a sketch of the curve.

Ans -5



Q 4 *Verify Stoke's theorem* $\overline{F} = y\hat{i} + z\hat{j} + x\hat{k}$ *and Surface S is the portion of the sphere for* $x^2 + y^2 + z^2 = 1$ *above the xy-Plane.*

Ans: $-\pi$