Unit-I: Matrices

Topics of Unit-I

- Symmetric, Skew-symmetric, Orthogonal Matrices
- Complex Matrices and Problems
- Inverse of Matrix by Using Elementary Transformations
- Introduction to Rank of Matrix and Rank of Matrix by Using Elementary Transformations

- Consistency of Non-Homogeneous System of Linear Equations
- Solution of Non-Homogeneous System of Linear Equations
- Solution of Homogeneous System of Linear Equations
- Linear Dependence and Independence of Vectors
- Eigen Values and Properties
- Definition of Eigen Vectors and Problems
- Problems on Eigen Vectors
- Cayley-Hamilton Theorem and its Application

LECTURE-1

- **SYMMETRIC MATRIX**
- **SKEW SYMMETRIC MATRIX**
- **ORTHOGONAL MATRIX**

SYMMETRIC MATRIX

A square matrix will be called symmetric, if for all values of i and j, $a_{ij} = a_{ji}$ i.e., $A' = A$

$$e.g., \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

SKEW - SYMMETRIC MATRIX

Skew Symmetric Matrix. A square matrix is called skew symmetric matrix, if
(1) $a_{ij} = -a_{ji}$ for all values of i and j, or $A' = -A$
(2) All diagonal elements are zero, *e.g.*,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

Properties of Symmetric Matrix

- Addition and difference of two symmetric matrices results in symmetric matrix.

- If A and B are two symmetric matrices and they follow the commutative property, i.e. AB =BA, then the product of A and B is symmetric.

- If matrix A is symmetric then A^n is also symmetric, where n is an integer.

- If A is a symmetric matrix then A^{-1} is also symmetric.

Properties of Skew-Symmetric Matrix

- When we add two skew-symmetric matrices then the resultant matrix is also skew-symmetric.
- Scalar product of skew-symmetric matrix is also a skew-symmetric matrix.
- The diagonal of skew-symmetric matrix consists of **zero elements** and therefore the sum of elements in the main diagonals is equal to zero.
- When identity matrix is added to skew-symmetric matrix then the resultant matrix is invertible.
- The determinant of skew-symmetric matrix is non-negative

Symmetric

$$A^T = \mathbf{A}$$

$$\begin{bmatrix} 9 & 2 & 3 \\ 2 & -1 & -8 \\ 3 & -8 & 0 \end{bmatrix}$$

Skew-Symmetric

$$A^T = -\mathbf{A}$$

$$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix}$$

Example :

Show that every square matrix can be uniquely expresses as the sum of a symmetric and a skew symmetric matrix.

Solution :

Let A be any square matrix

Evidently $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Taking $$P = \frac{1}{2}(A + A'),\ Q = \frac{1}{2}(A - A'), \text{ we get}$$

$$A = P + Q \qquad \dots(i)$$

Now $$P' = \frac{1}{2}[(A + A')]' = \frac{1}{2}\left[A' + (A')'\right]$$

$$= \frac{1}{2}(A' + A) = P$$

and $$Q' = \frac{1}{2}[(A-A')]' = \frac{1}{2}[A' - (A')']$$

$$= \frac{1}{2}(A' - A) = -Q$$

$$\Rightarrow \quad P' = P, Q' = -Q$$

Hence P is symmetric and Q is skew symmetric.

This shows that a square matrix A is expressible as a sum of a symmetric and skew symmetric matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Example :

Express the following matrix as the sum of a symmetric and a skew symmetric matrix,

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

Solution :

Given Matrix is A = $\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$

$$A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

Now,

$$B = \frac{1}{2}(A + A') = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

$$C = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$A = B + C = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

where B is symmetric and C is a skew-symmetric matrix.

ORTHOGONAL MATRIX

Orthogonal Matrix. A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix A' is an identity matrix *e.g.*,

$$A.\ A' = I$$

if $|A| = 1$, matrix A is proper.

Example :

Verify that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ *is orthogonal.*

Solution :

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \qquad \therefore \quad A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Verified.

Home Work

Q 1. Is given matrix is orthogonal. $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ **Ans . NO**

Q 2 Express given matrix A as sum of a symmetric and skew symmetric matrices.

$$A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix}.$$

$$\left[\textbf{Ans.} \begin{bmatrix} 6 & 6 & 3 \\ 6 & 2 & 5 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}\right]$$

Q.3 If A and B are two orthogonal matrix then show that the matrix AB and BA are orthogonal.

LECTURE-2

COMPLEX MATRIX

COMPLEX MATRIX

A matrix is called a complex matrix if at least one entry of the matrix is a complex number.

EXAMPLE

A = $\begin{bmatrix} 2i & 4-3i \\ 9 & 7 \end{bmatrix}$

Complex Number : $a + ib$

Conjugate of Complex Number:

$$\overline{a+ib} = a - ib$$

CONJUGATE OF A MATRIX

The matrix obtained from any given matrix A after replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and denoted by $\bar{A}$.

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 1+2i & 2-3i \\ 8 & 7 \end{bmatrix} \text{ then}$$

$$\bar{A} = \begin{bmatrix} 1-2i & 2+3i \\ 8 & 7 \end{bmatrix}.$$

CONJUGATE TRANSPOSE OF A MATRIX

Conjugate of a transpose is denoted as

$A^* = (\bar{A})^T = \overline{(A^T)} = A^\theta$

EXAMPLE

The conjugate transpose of a

A = $\begin{bmatrix} \mathbf{1+2}\boldsymbol{i} & \mathbf{2-3}\boldsymbol{i} \\ \mathbf{4-5}\boldsymbol{i} & \mathbf{5+6}\boldsymbol{i} \end{bmatrix}$ is

A* = $\begin{bmatrix} \mathbf{1-2}\boldsymbol{i} & \mathbf{4+5}\boldsymbol{i} \\ \mathbf{2+3}\boldsymbol{i} & \mathbf{5-6}\boldsymbol{i} \end{bmatrix}$.

HERMITIAN MATRIX

A square matrix A is called Hermitian matrix if $A^* = A$.

EXAMPLE

$$A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$$ is a Hermitian Matrix.

Note:

- $a_{ij} = \overline{a_{ji}}$
- Diagonal elements are all real.

SKEW -HERMITIAN MATRIX

A square matrix A is called skew-Hermitian matrix if $A^* = -A$.

EXAMPLE

A = $\begin{bmatrix} 3i & 1+i & 7 \\ -1+i & 0 & -2-i \\ -7 & 2-i & -i \end{bmatrix}$ is a skew-Hermitian Matrix.

Note:

- $a_{ij} = -\overline{a_{ji}}$
- Diagonal elements are Zero or purely imaginary number.

Show that every square matrix is expressible as the sum of a Hermitian matrix and a skew Hermitian matrix.

$$A = \frac{1}{2}[A + A^{\theta}] + \frac{1}{2}[A - A^{\theta}]$$

↓ Hermitian Matrix

↓ Skew-Hermitian Matrix

EXAMPLE :

Express the matrix $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ as a sum of Hermitian and skew-Hermitian matrix.

SOLUTION :

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix} \quad ...(1)$$

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \quad \Rightarrow \quad \overline{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix} \qquad ...(1)$$

$$(\overline{A})' = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix} \quad \text{or} \quad A^{\theta} = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix} \qquad ...(2)$$

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \qquad A^{\theta} = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

On adding (1) and (2), we get

$$A + A^{\theta} = \begin{bmatrix} 2 & 2-2i & 4-6i \\ 2+2i & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$$

On subtracting (

$$A - A^{\theta} = \begin{bmatrix} 2i & 2+2i & 6-4i \\ -2+2i & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$$

Let $$R = \frac{1}{2}(A + A^{\theta}) = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} \quad \text{...(3)}$$

Let $$S = \frac{1}{2}(A - A^{\theta}) = \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix} \quad \text{...(4)}$$

From (3) and (4), we have

$$A = \underbrace{\begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}}_{\text{Hermitian matrix}} + \underbrace{\begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}}_{\text{Skew-Hermitian matrix}} \qquad \text{Ans.}$$

EXAMPLE :

If $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$ *show that* $A^{\theta}A$ *is Hermitian matrix.*

SOLUTION : We have

$$A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

then

$$A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$$

Thus

$$A^{\theta} = (\bar{A}') = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$\therefore \quad A^{\theta}A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix} \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$= \begin{bmatrix} (2-i)(2+i)+(-5)(-5) & 3(2-i)+(-5)i & (2-i)(-1+3i)+(-5)(4-2i) \\ 3(2+i)+(-i)(-5) & 3\times 3+(-i)(i) & 3(-1+3i)+(-i)(4-2i) \\ (-1-3i)(2+i)+(4+2i)(-5) & 3(-1-3i)+i(4+2i) & (-1-3i)(-1+3i)+(4+2i)(4-2i) \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix}$$

We can observe here that $a_{ij} = \bar{a}_{ji}$ for all i and j. Hence, $A^{\theta}A$ is a Hermitian matrix.

UNITARY MATRIX

A square matrix A is called unitary matrix if $\boldsymbol{A\,A^{*} = I = A^{*}\,A}$.

EXAMPLE :

Following matrix is a unitary matrix:

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

EXAMPLE :

Under what condition the matrix $\mathbf{A} = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary.

SOLUTION :

$$\mathrm{A} = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}, \bar{A} = \begin{bmatrix} \alpha - i\gamma & -\beta - i\delta \\ \beta - i\delta & \alpha + i\gamma \end{bmatrix} \text{ and}$$

$$A^{*} = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

For a square matrix A to be unitary matrix, $\boldsymbol{A\,A^{*}= I = A^{*}A}$

$$A^{*}A=\begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$A^{*}\,A = \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

$$A\,A^{*} = \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

So, A to be unitary $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

Ques1:- If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ is a matrix, then show that $(I-N)(I+N)^{-1}$ is unitary matrix, where I is the Identity matrix.

[2012-13]

Soln:- we have, $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1-0 & -(1+2i) \\ 1-2i & 1-0 \end{bmatrix} \quad (1)$$

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$|I+N| = 6$

$$\text{Adj}(I+N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \Rightarrow (I+N)^{-1} = \frac{\text{Adj}(I+N)}{|I+N|} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \quad (2)$$

for unitary matrix, $A \cdot A^{\theta} = I = A^{\theta} \cdot A$

from (1) & (2) we get

$$(I-N)(I+N)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = B \text{ (say)}$$

Now $(\overline{B})^T = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = B^\theta$

$$\therefore \quad B^\theta \cdot B = \frac{1}{36}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \boxed{B^\theta \cdot B = I}$$

Hence Proved.

HOME WORK

Q1: Show that A = $\begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$ is Hermitian and *i*A is skew – Hermitian.

Q.2 Express $\begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$ as sum of hermitian and skew hermitian matrices.

$$\left[\textbf{Ans. } \frac{1}{2}\begin{bmatrix} -4 & 4-i & 4 \\ 4+i & 8 & 10 \\ 3-i & 2 & -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6i & -2-i & 2i \\ 2-i & -10i & 0 \\ -1+i & 2i & 4i \end{bmatrix}\right]$$

Q3: Show that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary

Q-4 Define Unitary matrix. Also Show that following matrix is unitary matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

LECTURE-3

Inverse of Matrix
by using
Elementary Transformations
(GAUSS-JORDAN METHOD)

Elementary Operations of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as *elementary operations* or *transformations*.

(i) *The interchange of any two rows or two columns*. Symbolically the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_2$ to $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \end{bmatrix}$, we get $\begin{bmatrix} -1 & \sqrt{3} & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}$.

(ii) *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$.

The corresponding column operation is denoted by $C_i \to kC_i$

For example, applying $C_3 \to \frac{1}{7}C_3$, to $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 & \frac{1}{7} \\ -1 & \sqrt{3} & \frac{1}{7} \end{bmatrix}$

(iii) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.*

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$.

The corresponding column operation is denoted by $C_i \to C_i + kC_j$.

For example, applying $R_2 \to R_2 - 2R_1$, to $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$.

Inverse of a Matrix

If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

INVERSE OF MATRIX BY E-OPERATIONS

(Gauss-Jordan Method)

By elementary operations on matrix we can find the inverse of the matrix, if the matrix is non-singular by the following procedure

We know that

$$A = AI \quad or \quad A = IA \qquad \ldots(1)$$

Now by the elementary row operations only we convert the above expression as

$$I = A^{-1}A$$

Where A^{-1} is the inverse of A

EXAMPLE :

Find the inverse of the matrix by Gauss-Jordan method of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

SOLUTION :

Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

And we know that $A = IA$

So put the value of A and I, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$by\ R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$by\ R_2 \to -R_2, R_1 \to R_1 + 2R_3$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 2 \\ 2 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix}1 & 0 & -3\\0 & 0 & 1\\0 & -1 & -3\end{bmatrix} = \begin{bmatrix}-5 & 0 & 2\\2 & -1 & 0\\-3 & 0 & 1\end{bmatrix} A$$

$by\ R_1 \to R_1 + 3R_2\ , R_3 \to R_3 + 3R_2$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & -1 & 0\end{bmatrix} = \begin{bmatrix}1 & -3 & 2\\2 & -1 & 0\\3 & -3 & 1\end{bmatrix} A$$

$by\ R_2 \to -R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -3 & 2\\-3 & 3 & -1\\2 & -1 & 0\end{bmatrix} A$$

$$I = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

EXAMPLE

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$.

SOLUTION

Let $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Applyirg $R_3 \rightarrow R_3 - 2R_1$, $R_4 \rightarrow R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + R_3$, we have

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, $R_4 \to R_4 - R_2$, we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, $R_4 \to R_4 - R_2$, we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$, $R_4 \to 2R_4 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_4$, $R_2 \to R_2 - R_4$, $R_3 \to R_3 - 3R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ 6 & -8 & 10 & -6 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

Applying $R_3 \to -\frac{1}{2}R_3$, $R_4 \to (-1)R_4$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

Hence

$$A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}.$$

Ques 11:- Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ [2020-21]

Soln:- write $A = IA$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -3 & -7 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (R_2 \to R_2 - 2R_1)$$

$$\begin{bmatrix} 1 & 3/2 & 2 \\ 0 & -3 & -7 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (R_1 \to R_1/2)$$

$$\begin{bmatrix} 1 & 3/2 & 2 \\ 0 & -3 & -7 \\ 0 & 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \quad (R_3 \to R_3 - R_1)$$

$$\begin{bmatrix} 1 & 3/2 & 2 \\ 0 & 1 & 7/2 \\ 0 & 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 2/3 & -1/3 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \quad (R_2 \to -\tfrac{1}{3} R_2)$$

$$\begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 7/3 \\ 0 & 0 & 5/6 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 2/3 & -1/3 & 0 \\ -5/6 & 1/6 & 1 \end{bmatrix} A$$

$(R_3 \to R_3 - \tfrac{1}{2} R_2)$

$$\begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 7/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 2/3 & -1/3 & 0 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$(R_3 \to \tfrac{6}{5} R_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$R_1 \to R_1 + \tfrac{3}{2} R_3$

$R_2 \to R_2 - \tfrac{7}{3} R_3$

$$A^{-1} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$$

HOME WORK

1. Find the inverse of the following matrices:

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 6 \end{bmatrix}.$$

$$\left[\textbf{Ans. } \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix}\right]$$

2. $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 3 & 8 \end{bmatrix}.$

$$\left[\textbf{Ans. } \frac{1}{130}\begin{bmatrix} 4 & -42 & 20 \\ 20 & 50 & -30 \\ -9 & -3 & 20 \end{bmatrix}\right]$$

3. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}.$

$$\left[\textbf{Ans. } \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}\right]$$

LECTURE-4

Rank of Matrix

and

Rank of Matrix by using Elementary Transformations

(Echelon Form)

DEFINITION

- Let A be any m×n matrix. It has square sub-matrices of different order
- The determinants of these square sub matrices are called minors of A.
- A matrix A is said to be of rank r if
- (i) It has at least one non zero minor of order r
- (ii) All the minor of order (r+1) or higher than r
- are zero.
- Symbolically , rank of A is written as $\rho(A) = r$

Some Results on Rank of the Matrix

(i) Only null matrix has zero rank.

(ii) If the order of an identity matrix is r then the rank of that matrix is also r. i.e the rank of unit matrix is same as of its order.

(iii) If A is a non zero, non singular matrix of order $n \times n$ then rank of A is $\rho(A) = n$.

(iv) If A is any matrix of order $m \times n$ then $\rho(A) \leq \min(m, n)$

(v) If all minors of order r are zero then $\rho(A) < r$.

METHODS OF FINDINGS RANK

To determine the rank of a matrix A, we adopt following different methods.

Method 01

- Start with the highest order minor of A. Let their order be r. If any one of them is non zero, then $\rho(A) = r$.
- If **all of them** are zero ,start with minors of next lower order (r-1) and so on till you get a non zero minor.
- The order of that minor is the rank of A.

Method 02

Echelon form method: In this form of matrix , each of the first 'r' elements of the leading diagonal is non zero and every elements below this diagonal / r^{th} row is zero. A matrix is reduced in echelon form as:

- The first non zero elements in row should be unity if possible.
- All the non zero rows, if any precede the zero rows.
- The rank of the matrix is equal to no. of non-zero diagonal elements or the no. of non zero rows when it has been reduced to Echelon form .

Continued.........

- In other words a matrix $A = [a_{ij}]$ is an Echelon matrix or is said to be in Echelon form if the no. of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain.
- In row reduced Echelon form (r).

❑ Echelon form method:

In this method the rank of the matrix is equal to **the no. of non-zero rows** when it has been reduced to **Echelon form.**

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Here, no. of non-zero rows is 2. So rank of the matrix = 2.

Numerical Problems

EXAMPLE:

Find the rank of the matrix. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Solution: $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

No of non zero rows =2

$\therefore$ hence rank of (A)= 2

EXAMPLE:

Use elementary transformations to reduce the following matrix A to triangular form and hence find the rank of A

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

The given matrix is $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \text{ by } R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \text{ by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 4 & 9 & 10 \end{bmatrix} \text{ by } R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_4 \to R_4 - R_3$$

$$\sim\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad by\ R_3 \to R_3 - \frac{4}{5}R_2$$

$$hence\ the\ number\ of\ non\ zero\ rows = 3,\ \text{so}\ \rho(A) = 3$$

EXAMPLE: Find the rank of the following matrix using Echelon method.

a) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Solution:

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$. (Applying $R_2 - 2R_1$ and $R_3 - R_1$)

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}. \text{ (Applying } R_3 + R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly rank $(A) = 2$

Solution:

(*b*) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$. (Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$)

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$. (Next, applying $R_3 \to R_3 - R_2$)

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence rank of A is **2**. **Ans.**

Solution:

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ (Applying here $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$)

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ Its rank $= 2$ since $\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} \neq 0$ $\therefore$ $\rho(A) = 2$

EXAMPLE:

Find the values of a and b such that the rank of matrix $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is 2.

SOLUTION:

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & -5 & a+3 & b-6 \end{bmatrix} \text{(on applying } R_3 \to R_3 - 3R_2 \text{ and } R_2 \to R_2 - 2R_1\text{)}$$

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & a-4 & b-6 \end{bmatrix} \text{(on applying } R_3 \to R_3 + R_2\text{)}$$

Since rank of A is given to be 2 we have $a = 4, b = 6$.

Home Work

Find the Echelon form of the following matrix and hence find the rank.

1. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$. [Ans. 3]

2. $\begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$. [Ans. 2]

3. Find rank by using Echelon form

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Ans. 2

Find the Echelon form and hence find the rank.

4. $$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

Ans. $$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}, \quad \text{rank} = 3$$

5. $$\begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$$

Ans. $$\begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{rank} = 2$$

6. $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & -3 & 4 \end{bmatrix}$

Ans. $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\quad \text{rank} = 3$

7. $\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$

Ans. $\begin{pmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & -11 & 10 & -15 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\quad \text{rank} = 2$

8. Find the value of P for which the matrix $A = \begin{pmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{pmatrix}$ is of rank 1.

LECTURE - 5

Rank of Matrix

Using

Elementary Transformations

(Normal Form)

• Method 3: **NORMAL FORM**

If A is an m x n matrix and by a series of elementary (row or column or both) operations, it can be put into one of the following forms (called Normal or Canonical forms):

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \qquad [I_r \quad 0], \qquad [I_r]$$

Where I_r is the unit matrix of the order r.

Since the rank of a matrix is not changed as a result of elementary transformations, it follows that $\rho(A) = r$

❑ Normal form method:

In this method the rank of the matrix is equal to the order of unit matrix. In this method both column and rows operations are used.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the order of unit matrix is 2.
So rank of the matrix = 2.

EXAMPLE:

Reduce the following matrix in to its normal form and hence find the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution: The given matrix is $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} \text{ by } R_3 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} by\, R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} by\, C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - 2C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 2 & 4 \\ 0 & 1 & 5 & 10 \end{bmatrix} by\, R_2 \leftrightarrow -R_2, R_3 \leftrightarrow -R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & 2 & 4 \end{bmatrix} \; by\; R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix} \; by\; R_3 \rightarrow R_3 - 6R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -56 \end{bmatrix} \; by\; C_3 \rightarrow C_3 - 5C_2, C_4 \rightarrow C_4 - 10C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \; by\; R_3 \leftrightarrow -\frac{1}{28} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; by \; C_4 \leftrightarrow -2C_3$$

$\sim [I_3 \quad \dots \quad o]$ Which is the normal form.

$$hence \; \rho(A) = 3$$

EXAMPLE:

Find the rank of the matrix

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}.$$ using normal form method.

SOLUTION:

$$A \sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix} \text{(on operating } C_{12} \text{ on A)}$$

$$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix} \text{(on operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 4R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix} \text{(on operating } C_2 \to C_2 - 6C_1, C_3 \to C_3 - 3C_1, C_4 \to C_4 - 8C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{(on operating } C_2 \to -\frac{1}{8} C_2 \text{ and then } C_4 \to C_4 - 17C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(on operating } R_3 \to R_3 - R_2 \text{ and } R_4 \to R_4 - R_2)$$

So, Rank(A)= 2

Ques 20:- Find the rank of $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$ by normal form.

[2019-20]

Solu:- $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$ $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 4C_1$, $C_4 \to C_4 - 2C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 3R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \leftrightarrow R_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C_3 \to C_3 - \frac{5}{7}C_2$, $C_4 \to C_4 - \frac{2}{7}C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to -\frac{1}{7}R_2$, $R_3 \to -R_3$

$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ which is a normal form.

$\Rightarrow \boxed{\rho(A) = 3}$.

Ques :- For non-singular matrices P and Q s.t. PAQ is normal Form $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

[2020-21]

Soln :- Let $A \sim I_3 A I_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - 2C_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad C_3 \to C_3 - C_2$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{Q} \Rightarrow \boxed{N = PAQ}$$

HOME WORK

Find rank by Normal form:

Q1. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$

Ans-1, rank of matrix = 2.

Q2. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Ans-2, rank of matrix =3.

Q3. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

Ans-3, rank of matrix = 3.

Q 4. Find rank by using Normal form

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix},$$

Q 5. Find rank by using Normal form

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

Q 6. Find non-singular matrices P & Q such that PAQ is in the normal form for the matrix hence find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{pmatrix}$$

Ans. P= $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, Q= $\begin{pmatrix} 1 & \frac{1}{3} & \frac{4}{15} & \frac{-1}{21} \\ 0 & \frac{-1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{-1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{pmatrix}$ and Rank of matrix $A = 2$

Lecture-6

Consistency of Non-Homogeneous System of Linear Equations

Solution of a System of Linear Equations

Let we consider the system of linear equations:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2 \qquad (1)$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

(Three equations with three unknowns)

Then the matrix form of the system of equations is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Or $$AX = B \qquad (2)$$

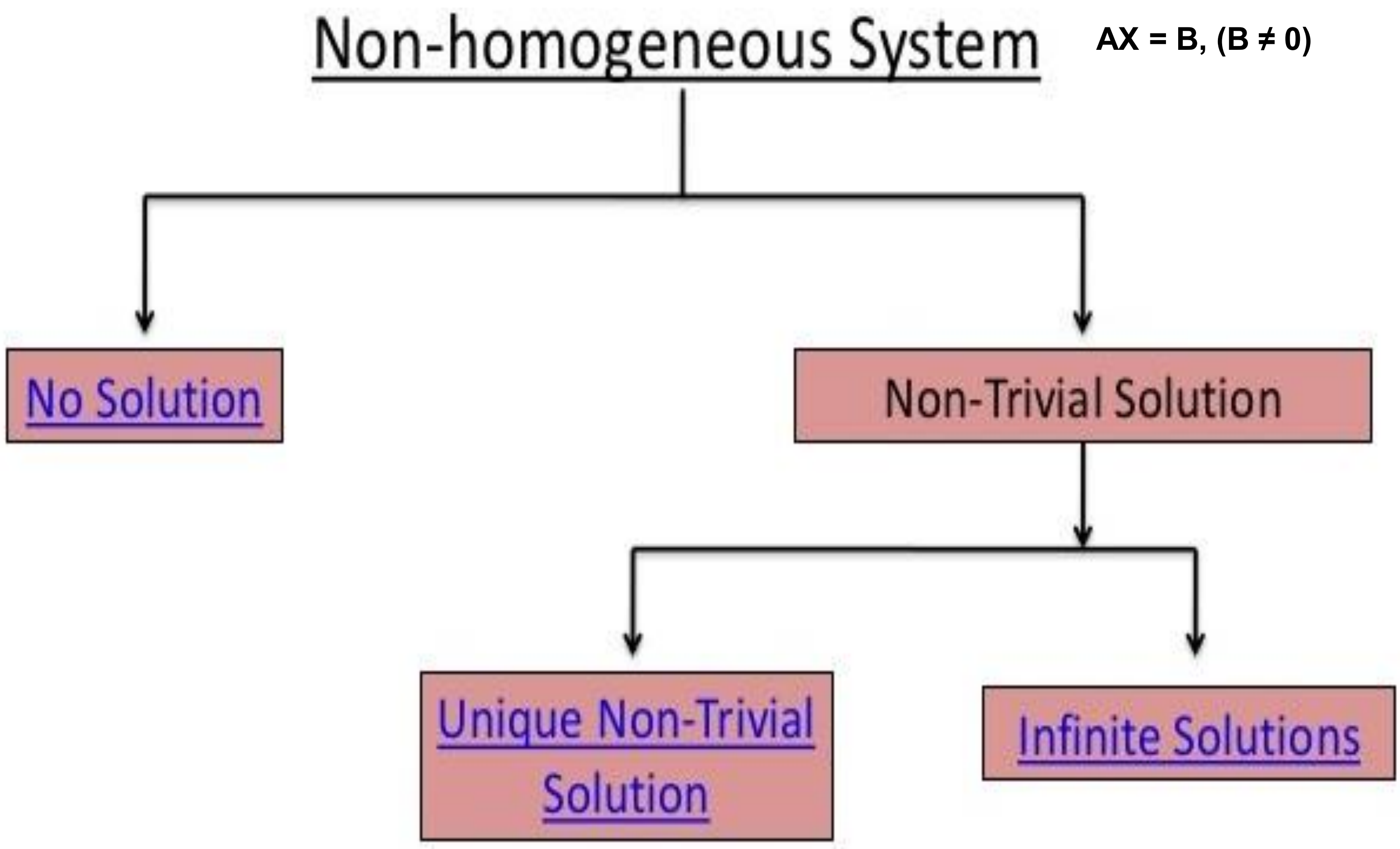
Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is said to be coefficient matrix.

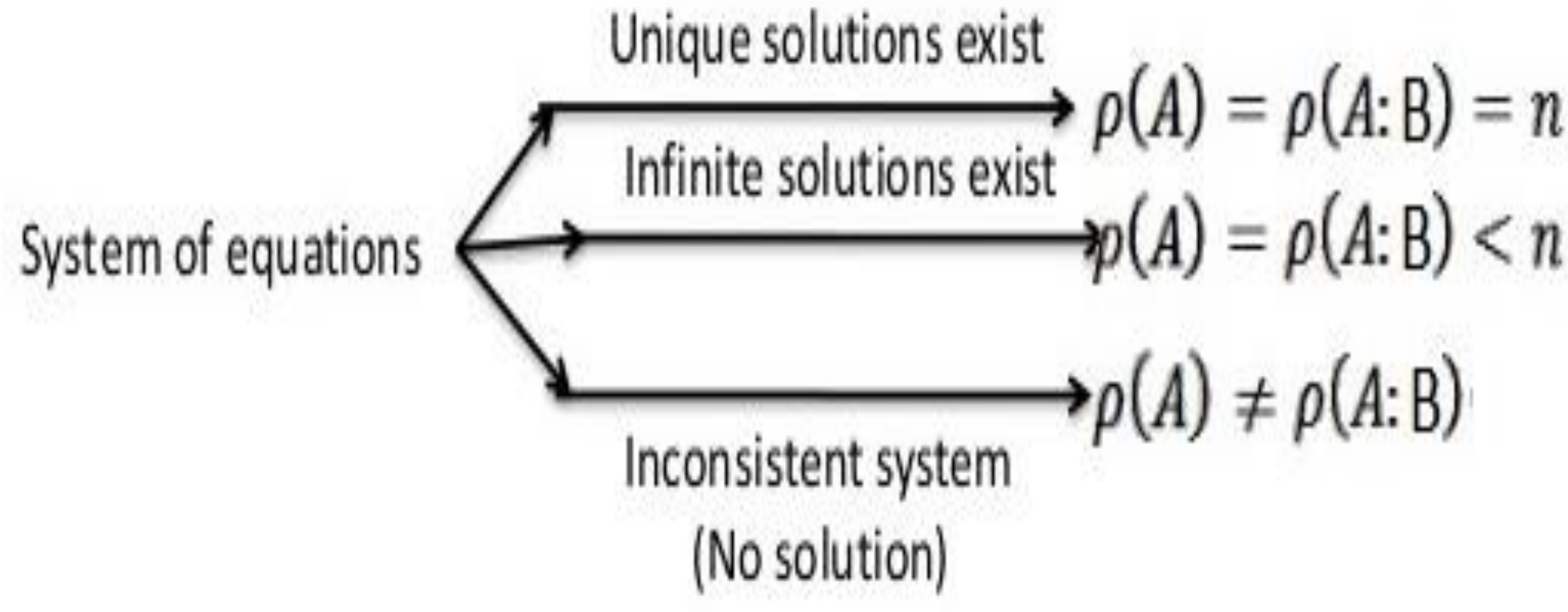
$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is said to be column vector or variable matrix.

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is said to be constant matrix.

If AX = B, (B = 0), then system is called homogenous system.

If AX = B, (B ≠ 0), then system is called non-homogenous system.





To check the consistency of the non homogeneous system of equations we form a matrix $C = [A : B]$, which is called augmented matrix and obtained by augmenting the elements of A & B or keeping the elements of A & B side by side.

EXAMPLE:

Solve, with the help of matrices, the simultaneous equations:

$x + y + z = 3,\ x + 2y + 3z = 4,\ x + 4y + 9z = 6.$

SOLUTION:

Augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 2 & 3 & \vdots & 4 \\ 1 & 4 & 9 & \vdots & 6 \end{bmatrix}$

Operating $R_{21}(-1), R_{31}(-1)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 3 & 8 & \vdots & 3 \end{bmatrix}$$

Operating $R_{32}(-3)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & 0 \end{bmatrix}$$

$\therefore \quad \rho[A : B] = 3. \quad$ Also $\rho(A) = 3.$

Since, $\rho[A : B] = \rho(A) = 3$ (number of unknowns).

Hence the given system of equations is consistent and has unique solution.

Equivalent system of equations is

$$x + y + z = 3$$

$$y + 2z = 1$$

$$2z = 0$$

$$\Rightarrow \quad x = 2, y = 1, z = 0.$$

EXAMPLE:

Show that the system of equations

$x + y + z = -3,\ 3x + y - 2z = -2,\quad 2x + 4y + 7z = 7$ *is not consistent.*

SOLUTION:

Augmented matrix,

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & -3 \\ 3 & 1 & -2 & \vdots & -2 \\ 2 & 4 & 7 & \vdots & 7 \end{bmatrix}$$

Operating $R_{21}(-3)$, $R_{31}(-2)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & -3 \\ 0 & -2 & -5 & \vdots & 7 \\ 0 & 2 & 5 & \vdots & 13 \end{bmatrix}$$

Operating $R_{32}(1)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & -3 \\ 0 & -2 & -5 & \vdots & 7 \\ 0 & 0 & 0 & \vdots & 20 \end{bmatrix}$$

which is echelon form.

$$\therefore \qquad \rho[A:B] = 3 \quad \text{and} \quad \rho(A) = 2$$

We observe that

$$\rho(A) \neq \rho(B)$$

Hence the system of given equations is inconsistent and has no solution.

EXAMPLE: *Investigate for consistency of the following equations:*

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

$$15x - 3y + 9z = 21.$$

SOLUTION:

Augmented matrix,

$$[A : B] = \begin{bmatrix} 4 & -2 & 6 & \vdots & 8 \\ 1 & 1 & -3 & \vdots & -1 \\ 15 & -3 & 9 & \vdots & 21 \end{bmatrix}$$

Operating R_{12}

$$\sim \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 4 & -2 & 6 & \vdots & 8 \\ 15 & -3 & 9 & \vdots & 21 \end{bmatrix}$$

Operating $R_{21}(-4)$, $R_{31}(-15)$

$$\sim \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 0 & -6 & 18 & \vdots & 12 \\ 0 & -18 & 54 & \vdots & 36 \end{bmatrix}$$

Operating $R_{32}(-3)$

$$\sim \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 0 & -6 & 18 & \vdots & 12 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\therefore \quad \rho[A : B] = 2 \quad \text{and} \quad \rho(A) = 2$$

Hence given system is consistent. But has infinite solutions.

$$x + y - 3z = -1 \qquad \text{...(1)}$$

$$-6y + 18z = 12 \qquad \text{...(2)}$$

Let z=k (arbitrary)

$$y = 3k - 2 \qquad \text{...From(2)}$$

$$x = 1 \qquad \text{...From(1)}$$

EXAMPLE:

Investigate, for what values of λ and μ do the system of equations

$$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + \lambda z = \mu$$

Have **(i)** No solution **(ii)** Unique solution **(iii)** Infinite solution

SOLUTION:

$$\text{Augmented matrix } [A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix}$$

Operating $R_{21}(-1), R_{31}(-1)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda - 1 & \vdots & \mu - 6 \end{bmatrix}$$

Operating $R_{32}(-1)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda-3 & \vdots & \mu-10 \end{bmatrix}$$

Case I. If $\lambda = 3, \mu \neq 10$

$$\rho(A) = 2, \rho[A:B] = 3$$

$\because$ $\rho(A) \neq \rho[A:B]$

$\therefore$ **The system has no solution.**

Case II. If $\lambda \neq 3$, μ may have any value

$$\rho(A) = \rho[A : B] = 3 = \text{number of unknowns.}$$

$\therefore$ The system has **unique solution.**

Case III. If $\lambda = 3$, $\mu = 10$

$$\rho(A) = \rho[A : B] = 2 < \text{number of unknowns.}$$

$\therefore$ The system has an **infinite number of solutions.**

Ques: Test the Consistency for the following system of equations and if system is consistent, solve them: [2022-23]

$$x+y+z=6,$$
$$x+2y+3z=14,$$
$$x+4y+7z=30.$$

Sol.^{n}: Given

$$x+y+z=6$$
$$x+2y+3z=14$$
$$x+4y+7z=30$$

Augmented Matrix $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 14 \\ 1 & 4 & 7 & : & 30 \end{bmatrix}$

operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get:

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 3 & 6 & : & 24 \end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$, we get:

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Rank $[A:B]$ = Rank $[A]$

$\therefore$ the system is consistent with infinite no. of solutions, and the equivalent system of equation is:

$x + y + z = 6$ ——— (1) and $y + 2z = 8$ ——— (2)

Let $z = k$

From (2), $y = 2(4-k)$

From (1), $x = 6 - 2(4-k) - k = k - 2$

$\therefore$ $\boxed{x = k-2}$, $\boxed{y = 2(4-k)}$, $\boxed{z = k}$

HOME WORK

1) Check the consistency of the following system of linear non-homogenous equations and find the solution, if it exists.

$7x_1 + 2x_2 + 3x_3 = 16;\ 2x_1 + 11x_2 + 5x_3 = 25,\ x_1 + 3x_2 + 4x_3 = 13$

(U.P.T.U. 2008)

Ans. $x_1 = \frac{95}{91},\ x_2 = \frac{100}{91},\ x_3 = \frac{197}{91}$

2) For what values of λ and μ, the following system of equations

$2x + 3y + 5z = 9,\ 7x + 3y - 2z = 8,\ 2x + 3y + \lambda z = \mu$

will have (i) unique solution and (ii) no solution

Ans. (i) $\lambda \neq 5$ (ii) $\mu \neq 9, \lambda = 5$

3) Determine the values of λ and μ for which the following system of equations

$3x - 2y + z = \mu,\ 5x - 8y + 9z = 3,\ 2x + y + \lambda z = -1$ has

(i) Unique solution (ii) No solution and

(iii) Infinite solutions.

Ans. (i) $\lambda \neq -3$ (ii) $\lambda = -3,\ \mu \neq \frac{1}{3}$ (iii) $\lambda = -3,\ \mu = 1/3$

Lecture-7

Problems Based Upon Non-Homogenous System of Linear Equations

Solution of a system of linear equations

EXAMPLE Solve

$$x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 3$$
$$4x_1 + 2x_2 - 2x_3 = 2.$$

SOLUTION:

By applying elementary row operations

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1, \\ R_3 \rightarrow R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2/(-3), \\ R_3 \rightarrow R_3/(-2) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 \to R_3 - R_2$$

$r(A) = 2 = r(A|B) < 3 = n =$ number of variables.

The system is consistent but has infinite number of solutions in terms of $n - r = 3 - 2 = 1$ variable. Choose $x_3 = k =$ arbitrary constant.

Solving $x_2 - x_3 = -1$ or $x_2 = x_3 - 1 = k - 1$.

$x_1 + x_2 - x_3 = 0$ or $x_1 = -x_2 + x_3 = -k + 1 + k = 1$

Thus the solutions are

$$x_1 = 1, x_2 = k - 1, x_3 = k, \quad \text{where } k \text{ is arbitrary.}$$

EXAMPLE

Determine the values of a and b for which the system

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

has (i) no solution (ii) unique solution (iii) infinite number of solutions. Find the solutions in case (ii) and (iii).

SOLUTION:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$$

$R_2 \to R_2 - R_1,$
$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right] \quad R_3 \to R_3 - R_2$$

Case 1: $a = 8, b \neq 15, r(A) = 2 \neq 3 = r(A|B)$, inconsistent, no solution.

Case 2: $a \neq 8, b$ any value, $r(A) = 3 = r(A|B)$ $= n =$ number of variables, unique solution, $z = \frac{b-15}{a-8}$,

$$y = (3a - 2b + 6)/(a - 8), x = z = (b - 15)/(a - 8).$$

Case 3: $a = 8, b = 15, r(A) = 2 = r(A|B) < 3 = n$, infinite solutions with $n - r = 3 - 2 = 1$ arbitrary variable. $x = k, y = 3 - 2k, z = k$, with k arbitrary.

EXAMPLE **Find the value of λ such that the following equations have unique solution:**

$$\lambda x + 2y - 2z - 1 = 0, \quad 4x + 2\lambda y - z - 2 = 0, \quad 6x + 6y + \lambda z - 3 = 0$$

And use matrix method to solve these equations when $\lambda = 2$.

(M.T.U. 2013)

SOLUTION: System has unique solution if coefficient matrix is Non-singular

$$\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0$$

$$(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0 \qquad \lambda \neq 2$$

when $\lambda = 2$.

$$[A : B] = \begin{bmatrix} 2 & 2 & -2 & \vdots & 1 \\ 4 & 4 & -1 & \vdots & 2 \\ 6 & 6 & 2 & \vdots & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & \vdots & 1/2 \\ 1 & 1 & -1/4 & \vdots & 1/2 \\ 1 & 1 & 1/3 & \vdots & 1/2 \end{bmatrix} \quad \begin{matrix} R_1 \Rightarrow R_1/(2), \\ R_2 \Rightarrow R_2/(4), \\ R_3 \Rightarrow R_3/(6) \end{matrix}$$

Operating $R_2 \Rightarrow R_2 - R_1$, $R_3 \Rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & \vdots & 1/2 \\ 0 & 0 & 3/4 & \vdots & 0 \\ 0 & 0 & 4/3 & \vdots & 0 \end{bmatrix}$$

Operating $R_2 \Rightarrow 4R_2/3$, $R_3 \Rightarrow 3R_3/4$

$$\sim \begin{bmatrix} 1 & 1 & -1 & \vdots & 1/2 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

$\therefore \quad \rho[A : B] = 2 = \rho(A) \; (< 3)$

Equivalent system of equations is

$$x + y - z = \frac{1}{2} \quad \text{and} \quad z = 0$$

$$\therefore \qquad x + y = \frac{1}{2}$$

Let $y = k_1$ then $x = \frac{1}{2} - k_1$

$$y = k_1$$

Ques :- Show that the system of eq^{n}s $3x+4y+5z=A$, $4x+5y+6z=B$, $5x+6y+7z=C$ are consistent only if A, B and C are in arithmetic progression. (AP). [2011-12]

Soln :- Let $AX=B$, be the given sys. of eq^{n}s

$$\Rightarrow C=[A:B]=\left[\begin{array}{ccc|c} 3 & 4 & 5 & A \\ 4 & 5 & 6 & B \\ 5 & 6 & 7 & C \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & A \\ 1 & 1 & 1 & B-A \\ 2 & 2 & 2 & C-A \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & B-A \\ 3 & 4 & 5 & A \\ 2 & 2 & 2 & C-A \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & B-A \\ 0 & 1 & 2 & 4A-3B \\ 0 & 0 & 0 & C-2B+A \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

which is echelon form

for consistency $\rho(A) = \rho[A \mid B]$

i.e. we must have $C - 2B + A = 0$

$$\Rightarrow C + A = 2B$$

$$\Rightarrow \boxed{B = \frac{C+A}{2}}$$

i.e. A, B and C are in A.P.

Hence Proved.

Ques: For what values of λ and μ, the system of linear eq^ns: $x+y+z=6$, $x+2y+5z=10$ and $2x+3y+\lambda z=\mu$ has (i) a unique sol^n (ii) No sol^n (iii) Infinite no. of sol^ns. Also, find the solution for $\lambda=2$ and $\mu=8$. [2019-20]

Sol^n: Let $AX=B$ be the system of eq^ns.

$$\Rightarrow C=[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 5 & : & 10\\ 2 & 3 & \lambda & : & \mu\end{bmatrix}$$

$R_2 \to R_2-R_1$ and $R_3 \to R_3-2R_1$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 4 & : & 4\\ 0 & 1 & \lambda-2 & : & \mu-12\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 4 & : & 4\\ 0 & 0 & \lambda-6 & : & \mu-16\end{bmatrix} \text{ --- (1)}$$

(i) Unique Sol^n

If $\rho(A)=\rho(C)=3$ then

$\lambda-6\neq 0 \Rightarrow \lambda\neq 6$ and $\mu-16\neq 0$

$\Rightarrow \boxed{\mu\neq 16}$ or μ may have any value

(ii) No Solution

If $\rho(A)\neq\rho(C) \Rightarrow \rho(A)=2$

$\Rightarrow \lambda-6=0$ and $\mu-16\neq 0$

$\Rightarrow \boxed{\lambda=6}$ and $\boxed{\mu\neq 16}$

(iii) Infinite no. of solutions

If $\rho(A)=\rho(C)=2$ then

$\lambda-6=0$ and $\mu-16=0$

$\Rightarrow \boxed{\lambda=6}$ and $\boxed{\mu=16}$

Now, substituting $\lambda=2$ and $\mu=8$ in eq^n (1), we get:

$$\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 4 & : & 4\\ 0 & 0 & -4 & : & -8\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 4\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 4\\ -8\end{bmatrix}$$

$x+y+z=6$ --- (2)

$y+4z=4$ --- (3)

$-4z=-8 \Rightarrow \boxed{z=2}$

From (3), $y+8=4$

$\Rightarrow \boxed{y=-4}$

From (2), $x-4+2=6$

$\Rightarrow \boxed{x=8}$

Hence, sol^n is –

$\boxed{x=8}$, $\boxed{y=-4}$ & $\boxed{z=2}$

HOME WORK

1. (*i*) Test the consistency of the following system of equations:

   $5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 11z = 5.$

   (*ii*) Test for the consistency of the following system of equations:

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 14 \\ 19 \end{bmatrix}$$

   (*iii*) Show that the equations $2x + 6y + 11 = 0,\ 6x + 20y - 6z + 3 = 0$ and $6y - 18z + 1 = 0$ are not consistent. (*U.K.T.U. 2011*)

2. Solve the following system of equations by matrix method:

   (*i*) $x + y + z = 8,\ x - y + 2z = 6,\ 3x + 5y - 7z = 14$

   (*ii*) $x + y + z = 6,\ x - y + 2z = 5,\ 3x + y + z = 8$

   (*iii*) $x + 2y + 3z = 1,\ 2x + 3y + 2z = 2,\ 3x + 3y + 4z = 1.$

ANSWERS

1. (i) Consistent

(ii) Consistent with many solutions.

1. (i) $x = 5, y = 5/3, z = 4/3$

(ii) $x = 1, y = 2, z = 3$

(iii) $x = -3/7, y = 8/7, z = -2/7$.

Lecture-8

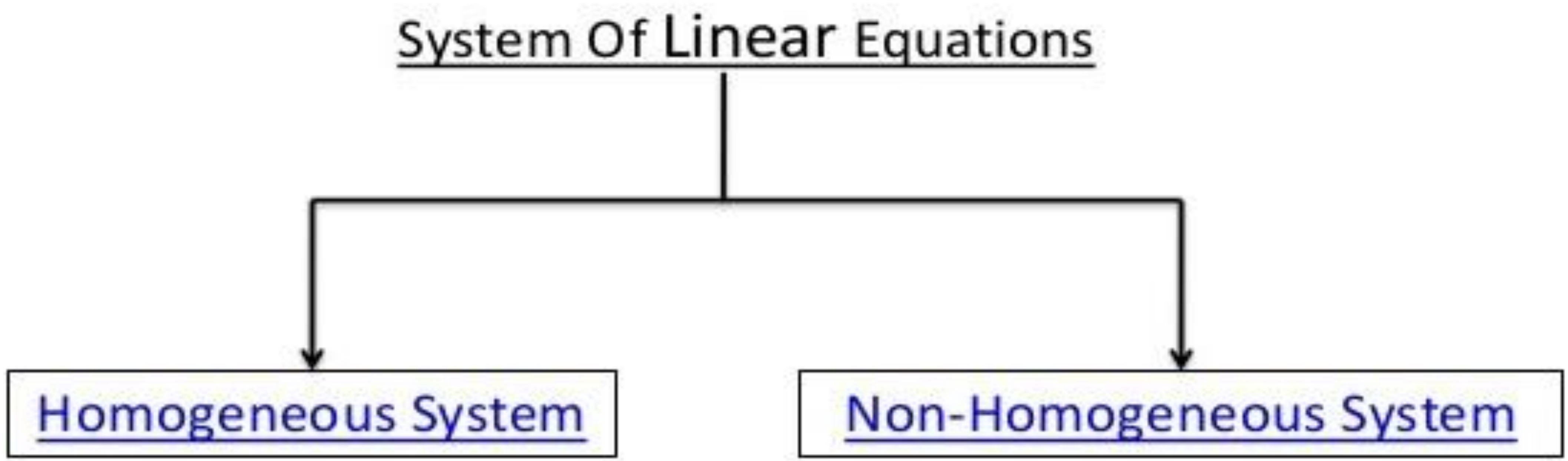
Solution of Homogenous System of Linear Equations

Linear System, Coefficient Matrix, Augmented Matrix

A **linear system of *m* equations in *n* unknowns** $x_1, \ldots, x_n$ is a set of equations of the following form

AX = B, (B = 0), then system is called ***homogenous system.***

AX = B, (B $\neq$ 0), then system is called non- homogenous system.



$AX = 0$, i.e., $(B = 0)$

$AX = B$, $(B \neq 0)$

Homogenous System (AX = 0, B = 0)

Unique Solution

If rank(A) = n (no. of unknowns, trivial solution)

Trivial Solution OR Zero solution

Infinite Solutions

If $\text{rank}(A) < n$ (no. of unknowns)

Non-Trivial Solution

EXAMPLE

Show that the equations $x + y + z = 0$, $2x + y - z = 0$, $x - 2y + z = 0$ have only the trivial solution.

SOLUTION:

The matrix form of the equations is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \quad X = O$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & -3 & 0 \end{pmatrix}$$

Applying $R_3 \to R_3 - 3R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$

Obviously,

$$\rho(A) = 3$$

The number of unknowns is 3.

Hence $\rho(A)$ = the number of unknowns.

$\therefore$ The equations have only the trivial solution.

EXAMPLE

Show that the equations $3x + y + 9z = 0$, $3x + 2y + 12z = 0$, $2x + y + 7z = 0$ have non trivial solutions also.

SOLUTION:

The matrix form of the equations is

$$\begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \quad X = O$$

$$A = \begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \neq 0$$

$\therefore \ \rho(A) = 2$

The number of unknowns is 3.

Hence $\rho(A) <$ the number of unknowns.

$\therefore$ The equations have non trivial solutions also.

Ques: Find the values of k for which the system of equations

$$(3k-8)x + 3y + 3z = 0$$
$$3x + (3k-8)y + 3z = 0$$
$$3x + 3y + (3k-8)z = 0$$

has a non-trivial solution. [AKTU-2022]

Sol^n: For the given system of equations to have a non-trivial solution, $\rho(A) < 3$, where

$$A = \begin{bmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{bmatrix}$$ is the Coefficient Matrix.

For this, $|A| = 0$.

$$\Rightarrow \begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

Operating $C_1 \rightarrow C_1 + C_2 + C_3$

$$\begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0 \Rightarrow (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0$$

Operating $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$

$$\Rightarrow (3k-2)(3k-11)^2 = 0$$

$$\Rightarrow \boxed{k = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}}$$ **Ans.**

EXAMPLE

Solve

$$
\begin{aligned}
x + y - 2z + 3w &= 0 \\
x - 2y + z - w &= 0 \\
4x + y - 5z + 8w &= 0 \\
5x - 7y + 2z - w &= 0.
\end{aligned}
$$

SOLUTION:

The coefficient matrix A is

$$
A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}
$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - 4R_1 \\ R_4 \longrightarrow R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \longrightarrow R_3 - R_1 \\ R_4 \longrightarrow R_4 - 4R_2 \end{matrix}$$

$r(A) = 2 < 4 = n =$ number of vari ables.

Non-trivial solutions exist in terms of $n - r = 4 - 2 = 2$ variables.
Choose $z = k_1$, and $w = k_2$. Then solving

$$x + y - 2z + 3w = 0$$
$$3y - 3z + 4w = 0$$

We get

$$y = \frac{1}{3}(3z - 4w) = z - \frac{4}{3}w = k_1 - \frac{4}{3}k_2$$
$$x = -y + 2z - 3w = -k_1 + \frac{4}{3}k_2 + 2k_1 - 3k_2$$
$$x = k_1 - \frac{5}{3}k_2$$

where k_1 and k_2 are arbitrary constants.

EXAMPLE *Find values of* λ

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$

$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$

$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0.$$

If system is consistent and have Non-trivial solution .Also find solution.

SOLUTION: System has Non-Trivial solution

$\rho(A) < 3$ where

$$A = \begin{bmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{bmatrix}$$

For this,

$$|A| = 0$$

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0$$

Operating $R_1 \rightarrow R_1 - R_2$

$$\begin{vmatrix} 0 & -\lambda + 3 & \lambda - 3 \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3\lambda - 3 \end{vmatrix} = 0$$

Operating $C_2 \rightarrow C_2 + C_3$

$$\begin{vmatrix} 0 & 0 & \lambda - 3 \\ \lambda - 1 & 5\lambda + 1 & \lambda + 3 \\ 2 & 6\lambda - 2 & 3\lambda - 3 \end{vmatrix} = 0$$

$\Rightarrow$ $(\lambda - 3)\,[(\lambda - 1)(6\lambda - 2) - 2(5\lambda + 1)] = 0$

$\Rightarrow$ $\lambda = 0,\ 3$

Putting $\lambda = 0$, we get

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-x + y = 0$$

$$-3y + 3z = 0$$

Let $z = k_1$ then from (2), $y = k_1$

From (1), $x = k_1$

$\therefore$ Infinite solutions are given by $x = k_1, y = k_1, z = k_1$ where k_1 is arbitrary.

Putting $\lambda = 3$, we get

$$A = \begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 10 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

Equivalent system of equations is

$$2x + 10y + 6z = 0$$

Let $z = k_2, y = k_3$ then from (3),

$$x = -5k_3 - 3k_2$$

Hence infinite solutions are given by

$$x = -5k_3 - 3k_2, \; y = k_3, \; z = k_2$$

HOME WORK

1.

$$x + y - 3z + 2w = 0,$$
$$2x - y + 2z - 3w = 0,$$
$$3x - 2y + z - 4w = 0,$$
$$-4x + y - 3z + w = 0.$$

Ans. Trivial solution $x = y = z = 0$, since $r(A) = 4 = n$

2.

$$x_1 + x_2 + x_3 + x_4 = 0,$$
$$x_1 + 3x_2 + 2x_3 + 4x_4 = 0,$$
$$2x_1 + x_3 - x_4 = 0.$$

Ans. $x_1 = -\frac{1}{2}k_1 + \frac{1}{2}k_2$, $x_2 = -\frac{1}{2}k_1 \frac{-3}{2}k_2$, $x_3 = k_1$, $x_4 = k_2$ where k_1 and k_2 are arbitrary constants giving infinite number of solutions: $r(A) = 2$, $n = 4$

HOME WORK

3. Determine the values of b for which the system of equations has non-trivial solutions. Find them.

$$(b-1)x + (4b-2)y + (b+3)z = 0,$$

$$(b-1)x + (3b+1)y + 2bz = 0,$$

$$2x + (3b+1)y + 3(b-1)z = 0.$$

Ans. **i.** $b = 0,\ x = y = z$

ii. $b = 3,\ x = -5k_1 - 3k_2,\ y = k_1,\ z = k_2$

where k_1 and k_2 are arbitrary

HOME WORK

4. Find the values of b for which the system has non-trivial solutions. Find them

$$
\begin{aligned}
2x + 3by + (3b+4)z &= 0,\\
x + (b+4)y + (4b+2)z &= 0,\\
x + 2(b+1)y + (3b+4)z &= 0.
\end{aligned}
$$

Ans. **i.** $b \neq \pm 2$, only trivial solution $x = y = z = 0$

ii. $b = 2$, $x = 0$, $z = k$, $y = -5k/3$, k arbitrary

iii. $b = -2$, $x = 4k$, $y = z = k$, k arbitrary.

LECTURE-10

Eigen values
and
its Properties

Eigen values of a square matrix

Let A be any square matrix of the order $n \times n$, I be the identity matrix of the same order and λ is any parameter then we have the following definitions:

(1) **Characteristic matrix**: The matrix $A - \lambda I$ is called the characteristic matrix of A, where

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \vdots & a_{2n} \\ \vdots & \ddots & \vdots & \ddots \\ a_{n1} & a_{n2} & \vdots & a_{nn} - \lambda \end{bmatrix}$$

(2) **Characteristic polynomial**: The determinant of the characteristic matrix that is $|A - \lambda I|$ is called the characteristic polynomial.

(3) **Characteristic equation**: The equation $|A - \lambda I| = 0$ is called the characteristic equation.

(4) **Eigen values or Characteristic values or latent roots of a square matrix**: The roots of the characteristic equation are known as the Eigen values of that matrix.

For example; $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$

$$= (2-\lambda)(6-5\lambda+\lambda^2-2)-2(2-\lambda-1)+1(2-3+\lambda) = -\lambda^3+7\lambda^2-11\lambda+5$$

Characteristic Equation: The equation $|A-\lambda I|=0$ is called the characteristic equation of the matrix A *e.g.*

$$\lambda^3-7\lambda^2+11\lambda-5=0$$

Characteristic Roots or Eigen Values: The roots of characteristic equation $|A-\lambda I|=0$ are called characteristic roots of matrix A. *e.g.*

$$\lambda^3-7\lambda^2+11\lambda-5=0$$

$\Rightarrow \quad (\lambda-1)(\lambda-1)(\lambda-5)=0 \qquad \therefore \lambda = 1, 1, 5$

Characteristic roots are 1, 1, 5.

Remember :

- Characteristic Equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

- No. of eigen values is equal to order of matrix.

Method to form cubic equation

λ^3 - (Trace of A) λ^2+ $(M_{11} + M_{22} + M_{33})$ λ - $|A|$ = 0

Where Trace of A = Sum of Diagonal Elements of A

& M_{11}, M_{22} and M_{33} are the respective minors of the matrix A

Some Important Properties of Eigen Values

(1) Any square matrix A and its transpose A' have the same eigen values.

Note. The sum of the elements on the principal diagonal of a matrix is called the **trace** of the matrix.

(2) The sum of the eigen values of a matrix is equal to the **trace** of the matrix.

(3) The product of the eigen values of a matrix A is equal to the **determinant** of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then the eigen values of

(*i*) kA are $k\lambda_1,\ k\lambda_2,\ \ldots..,\ k\lambda_n$

(*ii*) A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots.., \lambda_n^m$

(*iii*) A^{-1} are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \ldots.., \dfrac{1}{\lambda_n}$.

Applications of Eigen Values

(1).Eigen values are used in electric circuits, quantum mechanics, control theory, etc.

(2).They are used in the design of car stereo systems.

(3).They are also used to design bridges.

(4).It is not surprising to know that Eigen values are also used in determining Google's page rank.

(5).They are used in geometric transformations.

Question 1: Find the Eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: The characteristic polynomial of the given matrix is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5-\lambda & 2 & 1 \\ 5-\lambda & 3-\lambda & 1 \\ 5-\lambda & 2 & 2-\lambda \end{vmatrix} \textit{by operating } C_1 \to C_1 + C_2 + C_3$$

$$= (5-\lambda)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} \textit{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$= (5-\lambda)(1-\lambda)^2$$

so the characteristic equation of

$(5-\lambda)(1-\lambda)^2 = 0$ *so eigen values of A are* $\lambda = 1,1,5$

Question 2: Find the characteristic roots of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution: The characteristic polynomial of the given matrix is

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 8-\lambda & 0 & 2 \\ -6 & -5-\lambda & -4 \\ 2 & 5-3\lambda & 3-\lambda \end{vmatrix} \textit{by operating } C_2 \to C_2 + 3C_3$$

on Expanding the determinant we get

$$|A - \lambda I| = -\lambda^3 + 18\lambda^2 - 45\lambda = -\lambda(\lambda - 3)(\lambda - 15)$$

and the characteristic equation of matrix A is $|A - \lambda I| = 0$

Or $-\lambda(\lambda - 3)(\lambda - 15) = 0$ *so* $\lambda = 0, 3, 15$ these are the Eigen values of the matrix A.

Example 3. (*i*) *Obtain the eigen value of* A^3 *where* $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

(*ii*) *Two eigen values of the matrix* $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *are equal to 1 each. Find the eigen values of* A^{-1}.

Sol. (*i*) Characteristic eqn. of A is

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad \lambda = 1, 4$$

So, the eigen values of A^3 are 1^3, 4^3 *i.e.,* 1, 64.

(*ii*) Let λ be the third eigen value then,

$$\lambda + 1 + 1 = 2 + 3 + 2$$

$$\Rightarrow \qquad \lambda = 5$$

The three eigen values of A are 1, 1, 5.

$\therefore$ The eigen values of A^{-1} are $1, 1, \frac{1}{5}$.

Example 4. *The matrix A is defined as* $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find the eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Solution. $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)(3-\lambda)(-2-\lambda) = 0$ or $\lambda = 1, 3, -2$

Eigen values of $A^3 = 1, 27, -8$; Eigen values of $A^2 = 1, 9, 4$

Eigen values of $A = 1, 3, -2$; Eigen values of $I = 1, 1, 1$

$\therefore$ Eigen values of $3A^3 + 5A^2 - 6A + 2I$

First eigen value $= 3(1)^2 + 5(1)^3 - 6(1) + 2(1) = 4$

Second eigen value $= 3(27) + 5(9) - 6(3) + 2(1) = 110$

Third eigen value $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

Required eigen values are 4, 110, 10

Practice Questions

Q 1

Two of the eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ *are 3 and 6. Find the eigen values of* A^{-1}.

Ans : 1/2 , 1/3, 1/6

Q 2

Find the sum and product of the eigen values of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Ans : 6, 6

Q3. *Find the sum and product of the eigen values of the matrix*

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Ans: -3, 4

Q4. Find the eigenvalues of $3A^3 + 5A^2 - 6A + 2I$ where $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 3 & .2 \\ 0 & 0 & -2 \end{bmatrix}$

Ans: 4, 110, 10

LECTURE-11

Definitions of Eigen vectors and Problems

Eigen Vectors

Eigenvectors: If λ is the Eigen value of any n X n matrix A then the non-zero solution of

$$AX = \lambda X$$

Or $(A - \lambda I)X = 0$

is known as the Eigen vector corresponding to the Eigen value λ.

Properties of Eigenvectors: If X is the Eigen vector of a matrix A corresponding to the Eigen value λ then we have the following properties of Eigen vectors:

(i) Eigen vector is always a non-zero vector.

(ii) There may be more than one Eigen vectors corresponding to the same Eigen value of a Matrix.

(iii) Eigen vectors corresponding to the different Eigen values are linearly independent.

(iv) Eigen vectors corresponding to the same Eigen values may be linearly dependent.

(v) Eigen vectors of a symmetric matrix are orthogonal.

Example . *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Sol. The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$\lambda^2 - 5\lambda - 6 = 0$$

$\Rightarrow$ $$\lambda = 6, -1.$$

Thus, the eigen values of A are 6, – 1.

Corresponding to $\lambda = 6$, the eigen vectors are given by

$$(A - 6I)X_1 = 0$$

or $$\begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

or $$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get only one independent equation $-5x_1 - 2x_2 = 0$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-5} = k_1 \text{ (say)}$$

$$x_1 = 2k_1, \quad x_2 = -5k_1$$

$\therefore$ The eigen vectors are $X_1 = k_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Corresponding to $\lambda = -1$, the eigen vectors are given by

$$(A + I) X_2 = O$$

$$\Rightarrow \quad \begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - x_2 = 0$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{1} = k_2 \text{ (say)}$$

$$x_1 = k_2, \; x_2 = k_2$$

$\therefore$ The eigen vectors are $X_2 = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

EXAMPLE:

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

SOLUTION:

The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

or $(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$

$\therefore \lambda = 3, 2, 5$

Thus the eigenvalues of the given matrix are

$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5$

The eigenvectors of the matrix A corresponding to $\lambda = 2$ is

$[A - \lambda_1 I] X = 0$

i.e. $$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x_1 + x_2 + 4x_3 = 0$

$6x_3 = 0 \Rightarrow x_3 = 0$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2 = k_1$ (say) , $k_1 \neq 0$

Thus, the corresponding vector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The eigenvector corresponding to eigenvalue $\lambda_2 = 3$

$[A - \lambda_2 I]\, X = 0$

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x_2 + 4x_3 = 0$

$-x_2 + 6x_3 = 0$

and $2x_3 = 0 \Rightarrow x_3 = 0$

i.e. $x_2 = 0$ $\quad (\because x_3 = 0)$

Now let $x_1 = k_2$, we get the corresponding eigenvector as

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Again when $\lambda = 5$, the eigenvector is given by

$[A - \lambda_3 I] X = 0$

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + x_2 + 4x_3 = 0$

$-3x_2 + 6x_3 = 0$

or $x_2 = 2x_3 = k_3$ (say), $k_3 \neq 0$

Then

$2x_1 = x_2 + 4x_3 = k_3 + 2k_3$

$= 3k_3$

$$x_1 = \frac{3}{2}k_3$$

Thus, the corresponding vector is

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}k_3 \\ k_3 \\ \frac{1}{2}k_3 \end{bmatrix} = \frac{1}{2}k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

At Eigen Values $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5$

Eigen Vectors are

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \frac{1}{2}k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

EXAMPLE: Find eigen values and eigen vectors.

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

SOLUTION:

The characteristic equation of the given matrix is $|A - \lambda I| = 0$ i.e.

$$\begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$\lambda = 3, 4, -1$$

Now, eigenvectors corresponding to

$\Rightarrow \lambda = -1$ is

$[A - \lambda_1 I]\, X_1 = 0$

i.e. $\begin{bmatrix} 1+1 & 6 & 1 \\ 1 & 2+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $2x_1 + 6x_2 + x_3 = 0$

$x_1 + 3x_2 = 0$

$4x_3 = 0 \Rightarrow x_3 = 0$

$x_1 = -3x_2$

suppose $x_2 = k$, then $x_1 = -3k$, $k \neq 0$

The eigenvector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_2 = 3$ is

$[A - \lambda_2 I]X_2 = 0$

$$\text{i.e. } \begin{bmatrix} 1-3 & 6 & 1 \\ 1 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + 6x_2 + x_3 = 0$

$x_1 - x_2 = 0 \quad x_1 = x_2 = k$ (say), $k \neq 0$

$x_3 = 2x_1 - 6x_2 = 2k - 6k = -4k$

The eigenvector is

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ -4k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_3 = 4$ is

$[A - \lambda_3 I] X_3 = 0$

$$\begin{bmatrix} 1-4 & 6 & 1 \\ 1 & 2-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 + 6x_2 + x_3 = 0$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$-x_3 = 0 \Rightarrow x_3 = 0$

Let $x_2 = k$ then $x_1 = 2k$, Thus, the eigenvector is

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Example . *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

[U.K.T.U. 2011; G.B.T.U. (SUM) 2010]

Sol. The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

or
$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

or
$$(-2-\lambda)[-\lambda(1-\lambda)-12] - 2[-2\lambda-6] - 3[-4+1(1-\lambda)] = 0$$

or
$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By trial, $\lambda = -3$ satisfies it.

$\therefore \quad (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0 \Rightarrow (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0 \Rightarrow \lambda = -3, -3, 5$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda = -3$, the eigen vectors are given by

$$(A + 3I)\, X_1 = O$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

We get only one independent equation $x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = k_1$, $x_2 = k_2$ then $x_1 = 3k_1 - 2k_2$

$\therefore$ The eigen vectors are given by

$$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Corresponding to $\lambda = 5$, the eigen vectors are given by $(A - 5I) X_2 = O$

$$\Rightarrow \quad \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -7x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 - 2x_2 - 3x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

From first two equations,

$$\frac{x_1}{10 - 6} = \frac{x_2}{3 + 5} = \frac{x_3}{-2 - 2}$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k_3 \text{ (say)}$$

$$\therefore \quad x_1 = k_3, \; x_2 = 2k_3, \quad x_3 = -k_3$$

Hence the eigen vectors are given by

$$X_2 = k_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Example . *Find all the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution. Characteristic equation of A is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-2-\lambda)[-\lambda+\lambda^2-12]-2(-2\lambda-6)-3(-4+1-\lambda)=0$$

$$\lambda^3+\lambda^2-21\lambda-45=0 \qquad \text{....(1)}$$

By trial: If $\lambda=-3$, then $-27+9+63-45=0$, so $(\lambda+3)$ is one factor of (1).

The remaining factors are obtained on dividing (1) by $\lambda+3$.

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$\lambda^2-2\lambda-15=0 \qquad \Rightarrow \quad (\lambda-5)(\lambda+3)=0$

$\Rightarrow \quad (\lambda+3)(\lambda+3)(\lambda-5)=0 \;\Rightarrow\; \lambda=5,-3,-3$

To find the eigen vectors for corresponding eigen values, we will consider the matrix equation

$$(A-\lambda I)X=0 \qquad i.e., \quad \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots(2)$$

On putting $\lambda=5$ in eq. (2), it becomes

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $\quad -7x+2y-3z=0,$

$\qquad\qquad 2x-4y-6z=0$

$$\frac{x}{-12-12}=\frac{y}{-6-42}=\frac{z}{28-4} \quad \text{or} \quad \frac{x}{-24}=\frac{y}{-48}=\frac{z}{24} \quad \text{or} \quad \frac{x}{1}=\frac{y}{2}=\frac{z}{-1}=k$$

$$x=k, \quad y=2k, \quad z=-k$$

Hence, the eigen vector $X_1=\begin{bmatrix} k \\ 2k \\ -k \end{bmatrix}=k\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Put $\lambda=-3$ in eq. (2), it becomes

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $x+2y-3z=0,$

$$2x+4y-6z=0,$$

$$-x-2y+3z=0$$

Here first, second and third equations are the same.

Let $x=k_1$, $y=k_2$ then $z=\frac{1}{3}(k_1+2k_2)$

Hence, the eigen vector is $\begin{bmatrix} k_1 \\ k_2 \\ \frac{1}{3}(k_1+2k_2) \end{bmatrix}$

Let $k_1 = 0, k_2 = 3$, Hence $X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

Since the matrix is non-symmetric, the corresponding eigen vectors X_2 and X_3 must be linearly independent. This can be done by choosing

$k_1 = 3,\ k_2 = 0$, and Hence $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Hence, $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$

Ans.

HOME WORK

1) Find the eigen values and eigen vectors of the following matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Ans. Eigen values: 1, 1, 4; eigenvectors: $[1\ 1\ 0]^T$, $[-1\ 0\ 1]^T$, $[1\ -1\ 1]$

2) Find the eigen values and eigen vectors of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Ans. Eigenvalues: -2, 6, 3; Eigenvectors $[-1\ 0\ 1]^T$, $[1\ 2\ 1]^T$, $[1\ -1\ 1]$

3) Show that matrix $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ has less than three linearly independent vectors. Also find them.

Ans. Eigen values: 2, 2, 3; eigenvectors: $[5\ \ 2-5]^T$, $[1\ 1\ -2]^T$.

LECTURE-12

Cayley- Hamilton Theorem
and
Its Applications

Statement

Every square matrix satisfies its own characteristic equation.

Example 1

Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Also find A^{-1}.

Solution

- The characteristic equation of A is: $|A-\lambda I|=0$

or $$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$ or $$\lambda^3-6\lambda^2+9\lambda-4=0$$

By Cayley Hamilton theorem, we get $A^3-6A^2+9A-4I=0$

Here $$A^2=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$ and $$A^3=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3-6A^2+9A-4I=0$$

Hence proved

Verification:

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{pmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+5+5 & -6-10+5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4I$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Inverse of Matrix A,

$$A^3 - 6A^2 + 9A - 4I = 0$$

On multiplying by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0 \quad \text{or} \quad 4A^{-1} = A^2 - 6A + 9I$$

or $$4A^{-1} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix}, \quad A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

Example 2

Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Compute A^{-1} also find the matrix represented

By:

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

Characteristic equation of the matrix A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[(1-\lambda)(2-\lambda)]-1(0)+1(0-1+\lambda)=0 \quad \Rightarrow \quad \lambda^3-5\lambda^2+7\lambda-3=0$$

According to Cayley-Hamilton Theorem

$$A^3-5A^2+7A-3I=0 \qquad \ldots(1)$$

We have to verify the equation (1).

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Inverse of A

Pre multiplying (1) by A^{-1}

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

We get,

$$A^2 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I).$$

From (2),

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$

$$= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + A\left(A^3 - 5A^2 + 7A - 3I\right) + A^2 + A + I$$

$$= A^5 \times O + A \times O + A^2 + A + I \quad = A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2+1 & 4+1+0 & 4+1+0 \\ 0+0+0 & 1+1+1 & 0+0+0 \\ 4+1+0 & 4+1+0 & 5+2+1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

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Ques :- Verify Cayley-Hamilton th. for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and hence find A^{-1}.

[2019-20]

Soln :- $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - (\text{tr. of } A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

Now, for CHT

$A^2 = A \times A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16+0+1 & 0+0+0 & 4+0+1 \\ 0+0+2 & 0+1+0 & 0+2+2 \\ 4+0+1 & 0+0+0 & 1+0+1 \end{bmatrix}$

$= \begin{bmatrix} 17 & 0 & 5 \\ 2 & 1 & 4 \\ 5 & 0 & 2 \end{bmatrix}$

$$A^3 = A^2 \times A = \begin{bmatrix} 17 & 0 & 5 \\ 2 & 1 & 4 \\ 5 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 73 & 0 & 22 \\ 12 & 1 & 8 \\ 22 & 0 & 7 \end{bmatrix}$$

verification:- $A^3 - 6A^2 + 8A - 3I$

$$\begin{bmatrix} 73 & 0 & 22 \\ 12 & 1 & 8 \\ 22 & 0 & 7 \end{bmatrix} - 6\begin{bmatrix} 17 & 0 & 5 \\ 2 & 1 & 4 \\ 5 & 0 & 2 \end{bmatrix} + 8\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \implies \text{CHT is verified.}$$

for A^{-1}, pre-multiplication of A^{-1} on both sides of CHT

$$A^{-1}(A^3 - 6A^2 + 8A - 3I) = 0$$

$$\implies A^2 - 6A + 8I - 3A^{-1} = 0$$

$$A^{-1} = \frac{1}{3}(A^2 - 6A + 8I)$$

$$\implies \frac{1}{3}\left[\begin{bmatrix} 17 & 0 & 5 \\ 2 & 1 & 4 \\ 5 & 0 & 2 \end{bmatrix} - 6\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & -8 \\ -1 & 0 & 4 \end{bmatrix} \text{ Ans.}$$

Home work

1. Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

2. Compute A^{-1} also find the matrix represented

By: $A^6 - 6A^5 + 9A^4 - 2A^3 + 12A^2 + 23A - 9I.$

Ans: $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}$

2. Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

3. Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$