

# Introduction and types of Relations

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Relation or Binary relation  $R$  from set  $A$  to  $B$  is a subset of  $A \times B$  which can be defined as

$$aRb \leftrightarrow (a,b) \in R \leftrightarrow R(a,b).$$

A Binary relation  $R$  on a single set  $A$  is defined as a subset of  $A \times A$ . For two distinct set,  $A$  and  $B$  with cardinalities  $m$  and  $n$ , the maximum cardinality of the relation  $R$  from  $A$  to  $B$  is  $mn$ .

## Domain and Range:

if there are two sets  $A$  and  $B$  and Relation from  $A$  to  $B$  is  $R(a,b)$ , then domain is defined as the set  $\{ a \mid (a,b) \in R \text{ for some } b \text{ in } B \}$  and Range is defined as the set  $\{ b \mid (a,b) \in R \text{ for some } a \text{ in } A \}$ .

## Types of Relation:

1. **Empty Relation:** A relation  $R$  on a set  $A$  is called Empty if the set  $A$  is empty set.
2. **Full Relation:** A binary relation  $R$  on a set  $A$  and  $B$  is called full if  $A \times B$ .
3. **Reflexive Relation:** A relation  $R$  on a set  $A$  is called reflexive if  $(a,a) \in R$  holds for every element  $a \in A$  .i.e. if set  $A = \{a,b\}$  then  $R = \{(a,a), (b,b)\}$  is reflexive relation.
4. **Irreflexive relation :** A relation  $R$  on a set  $A$  is called reflexive if no  $(a,a) \in R$  holds for every element  $a \in A$ .i.e. if set  $A = \{a,b\}$  then  $R = \{(a,b), (b,a)\}$  is irreflexive relation.
5. **Symmetric Relation:** A relation  $R$  on a set  $A$  is called symmetric if  $(b,a) \in R$  holds when  $(a,b) \in R$ .i.e. The relation  $R = \{(4,5), (5,4), (6,5), (5,6)\}$  on set  $A = \{4,5,6\}$  is symmetric.
6. **AntiSymmetric Relation:** A relation  $R$  on a set  $A$  is called antisymmetric if  $(a,b) \in R$  and  $(b,a) \in R$  then  $a = b$  is called antisymmetric.i.e. The relation  $R = \{(a,b) \rightarrow R \mid a \leq b\}$  is anti-symmetric since  $a \leq b$  and  $b \leq a$  implies  $a = b$ .
7. **Transitive Relation:** A relation  $R$  on a set  $A$  is called transitive if  $(a,b) \in R$  and  $(b,c) \in R$  then  $(a,c) \in R$  for all  $a,b,c \in A$ .i.e. Relation  $R = \{(1,2), (2,3), (1,3)\}$  on set  $A = \{1,2,3\}$  is transitive.
8. **Equivalence Relation:** A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive. i.e. relation

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$  on set  $A = \{1, 2, 3\}$  is equivalence relation as it is reflexive, symmetric, and transitive.

9. **Asymmetric relation:** Asymmetric relation is opposite of symmetric relation. A relation  $R$  on a set  $A$  is called asymmetric if no  $(b, a) \in R$  when  $(a, b) \in R$ .

**Important Points:**

1. Symmetric and anti-symmetric relations are not opposite because a relation  $R$  can contain both the properties or may not.
2. A relation is asymmetric if and only if it is both anti-symmetric and irreflexive.
3. **Number of different relation from a set with  $n$  elements to a set with  $m$  elements is  $2^{mn}$**

Ex:

if  $R = \{r_1, r_2, r_3, \dots, r_n\}$  and  $S = \{s_1, s_2, s_3, \dots, s_m\}$   
 then Cartesian product of  $R$  and  $S$  is:  
 $R \times S = \{(r_1, s_1), (r_1, s_2), (r_1, s_3), \dots, (r_1, s_n),$   
 $(r_2, s_1), (r_2, s_2), (r_2, s_3), \dots, (r_2, s_n),$   
 $\dots, \dots, \dots,$   
 $(r_n, s_1), (r_n, s_2), (r_n, s_3), \dots, (r_n, s_n)\}$

This set of ordered pairs contains  $mn$  pairs.  
 Now these pairs can be present in  $R \times S$  or can be absent.  
 So total number of possible relation =  $2^{mn}$

**4. Number of Reflexive Relations on a set with  $n$  elements :  $2^{n(n-1)}$ .**

A relation has ordered pairs  $(a, b)$ . Now  $a$  can be chosen in  $n$  ways and same for  $b$ . So set of ordered pairs contains  $n^2$  pairs. Now for a reflexive relation,  $(a, a)$  must be present in these ordered pairs. And there will be total  $n$  pairs of  $(a, a)$ , so number of ordered pairs will be  $n^2 - n$  pairs. So total number of reflexive relations is equal to  $2^{n(n-1)}$ .

**5. Number of Symmetric Relations on a set with  $n$  elements :  $2^{n(n+1)/2}$ .**

A relation has ordered pairs  $(a, b)$ . Now for a symmetric relation, if  $(a, b)$  is present in  $R$ , then  $(b, a)$  must be present in  $R$ .

In Matrix form, if  $a_{12}$  is present in relation, then  $a_{21}$  is also present in relation and As we know reflexive relation is part of symmetric relation.

So from total  $n^2$  pairs, only  $n(n+1)/2$  pairs will be chosen for symmetric relation. So total number of symmetric relation will be  $2^{n(n+1)/2}$ .

**6. Number of Anti-Symmetric Relations on a set with n elements:  $2^n \cdot 3^{n(n-1)/2}$ .**

A relation has ordered pairs (a,b). For anti-symmetric relation, if (a,b) and (b,a) is present in relation R, then  $a = b$ . (That means a is in relation with itself for any a). So for (a,a), total number of ordered pairs = n and total number of relation =  $2^n$ . if (a,b) and (b,a) both are not present in relation or Either (a,b) or (b,a) is not present in relation. So there are three possibilities and total number of ordered pairs for this condition is  $n(n-1)/2$ . (selecting a pair is same as selecting the two numbers from n without repetition) As we have to find number of ordered pairs where  $a \neq b$ . it is like opposite of symmetric relation means total number of ordered pairs =  $(n^2) - \text{symmetric ordered pairs}(n(n+1)/2) = n(n-1)/2$ . So, total number of relation is  $3^{n(n-1)/2}$ . So total number of anti-symmetric relation is  $2^n \cdot 3^{n(n-1)/2}$ .

**7. Number of Asymmetric Relations on a set with n elements :  $3^{n(n-1)/2}$ .**

In Asymmetric Relations, element a can not be in relation with itself. (i.e. there is no  $aRa \forall a \in A$  relation.) And Then it is same as Anti-Symmetric Relations.(i.e. you have three choice for pairs (a,b) (b,a)). Therefore there are  $3^{n(n-1)/2}$  Asymmetric Relations possible.

**8. Irreflexive Relations on a set with n elements :  $2^{n(n-1)}$ .**

A relation has ordered pairs (a,b). For Irreflexive relation, no (a,a) holds for every element a in R. It is also opposite of reflexive relation.

Now for a Irreflexive relation, (a,a) must not be present in these ordered pairs means total n pairs of (a,a) is not present in R, So number of ordered pairs will be  $n^2 - n$  pairs.

So total number of reflexive relations is equal to  $2^{n(n-1)}$ .

**9. Reflexive and symmetric Relations on a set with n elements :  $2^{n(n-1)/2}$ .**

A relation has ordered pairs (a,b). Reflexive and symmetric Relations means (a,a) is included in R and (a,b)(b,a) pairs can be included or not. (In Symmetric relation for pair (a,b)(b,a) (considered as a pair). whether it is included in relation or not) So total number of Reflexive and symmetric Relations is  $2^{n(n-1)/2}$ .